Analytic subvarieties with many rational points

C. Gasbarri

Abstract.We give a generalization of the classical Bombieri–Schneider–Lang criterion in transcendence theory. We give a local notion of LG–germ, which is similar to the notion of E– function and Gevrey condition, and which generalize (and replace) the condition on derivatives in the theorem quoted above. Let $K \subset \mathbb{C}$ be a number field and X a quasi–projective variety defined over K. Let $\gamma\colon M \to X$ be an holomorphic map of finite order from a parabolic Riemann surface to X such that the Zariski closure of the image of it is strictly bigger then one. Suppose that for every $p \in X(K) \cap \gamma(M)$ the formal germ of M near P is an LG– germ, then we prove that $X(K) \cap \gamma(M)$ is a finite set. Then we define the notion of conformally parabolic Khäler varieties; this generalize the notion of parabolic Riemann surface. We show that on these varieties we can define a value distribution theory. The complementary of a divisor on a compact Khäler manifold is conformally parabolic; in particular every quasi projective variety is. Suppose that A is conformally parabolic variety of dimension m over $\mathbb{C}$ with Khäler form ω and $\gamma\colon A \to X$ is an holomorphic map of finite order such that the Zariski closure of the image is strictly bigger then m. Suppose that for every $p \in X(K) \cap \gamma(A)$, the image of A is an LG–germ. then we prove that there exists a current T on A of bidegree $(1,1)$ such that $\int_A T \wedge \omega^{m-1}$ explicitly bounded and with Lelong number bigger or equal then one on each point in $\gamma^{-1}(X(K))$. In particular if A is affine $\gamma^{-1}(X(K))$ is not Zariski dense.

1 Introduction.

A classical theorem by Schneider and Lang (cf. [La]), asserts that if we have $N > 1$ meromorphic functions $f_1(z), \dots, f_N(z)$ which are of *finite order* and such that $\frac{df_i}{dz} \in K[f_1, \dots, f_n]$ for some number field K, then, if $Trdeg_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}(f_1 \dots, f_N) \geq 2$, the set of points $z \in \mathbb{C}$ such that $f_i(z) \in K$ for every i, is finite. This has been generalized by Bombieri (cf. [Bom] and [De1]): given N meromorphic functions $f_1, \dots, f_N$ on $\mathbb{C}^n$ such that $Trdeg_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}(f_1, \dots, f_N) \geq n+1$ and with a similar condition on the derivatives, then the set of points $\underline{z} \in \mathbb{C}^n$ such that $f_i(\underline{z})$ is defined and belongs to K for every i, is not Zariski dense in $\mathbb{C}^N$.

In the Schneider–Lang Theorem, the condition on the derivatives can be rephrased by saying that the image of $\mathbb{C}$ in $\mathbb{C}^n$ is the leaf of an algebraic foliation defined over K (and similarly in the higher dimensional case); on the other side the fact that the functions are algebraically independent means that this leaf is Zariski dense in an algebraic variety of dimension strictly bigger then one.

When one looks closely to the proof of the Schneider–Lang Theorem, one realize that the condition on the derivatives, namely that the ring $K[f_1, \dots, f_n]$ is closed under the derivation, is used only locally around the points. This condition is needed in order to derive suitable bounds on the successive derivatives of $P(f_1, \dots, f_n)$, where P is an "auxiliary polynomial".

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Besides, the notion of *map of finite order* from a variety M to another variety X make sense in a more general contest then the classical one appearing in the Bombieri–Schneider–Lang Theorems (where one deals with meromorphic functions on $\mathbb{C}^n$).

Actually there is a notion of maps of finite order as soon as we can define a it counting function $T_\gamma(r)$ á la Nevanlinna to any meromorphic map $\gamma\colon M \to X$. To achieve this it is natural to assume that X is holomorphically embedded into some complex projective space, or equivalently (after replacing X by its Zariski closure) that X is a projective variety. Then, when M is one dimensional, the counting function may be defined when the Riemann surface M is *parabolic* in the sense of the classical theory of Myrberg, Nevanlinna, Ahlfors; in particular, as soon as it is an an affine variety or a finite ramified covering of the complex plane. Moreover, the extension of Nevanlinna's theory by Griffiths and King (cf. [GK]) leads to natural notions of holomorphic maps of finite order from any higher dimensional *smooth affine* complex algebraic variety M to a complex projective variety X. We will see in §5 that we can define a value distribution theory for a wider class of varieties.

It is natural to ask if a general value distribution theory allows to develop a more general theory of algebraic values of analytic maps. In this paper we deal with such a generalization.

In the first part of the paper we give a general statement on formal germs around a rational point (over some number field K) on a projective variety: if some hypothesis are verified then such a germ is algebraic; these hypothesis are of "Arakelovian" nature: we ask that the norms (at finite and infinite places) of some maps between hermitian modules over the ring of integers of K are explicitly bounded. The benefit of this approach is that, in this way, it is easier to understand where the difficulties are localized: we can work on each place independently and then find some global arithmetic relations.

In the third section we deal with the finite places: we introduce the notion of LG–germ: given a (say, to simplify, one dimensional) smooth germ $\widehat{V}$ of analytic variety around a rational point P of an N–dimensional variety X, roughly speaking we say that it is an LG–germ of type α, if, in suitable choice of the coordinates around P, $\widehat{V}$ is given by N power series which are of the form $\sum \frac{a_i z^i}{(i!)^\alpha}$, with a_i in the ring of the S–integers of K, for some finite set S of places of K. The notion of LG–germ is exactly what is needed at finite places in order to let the statement in section 2 work. It is worth remarking that a leaf of a foliation in a smooth point is an LG–germ and to require that a function from $\mathbb{C}^n$ to $\mathbb{C}^N$ defines an LG–germ around K–rational points (of $\mathbb{C}^N$) is less demanding then to require the condition on the derivatives in the classical Bombieri–Schneider–Lang Theorem: essentially because the notion of being an LG–germ is very local and the condition on derivatives in the Bombieri–Schneider–Lang Theorem is global. The notion of LG–germ is similar to the notion of Gevrey functions and of E– function in the transcendence theory of values of the solutions of differential equations developed by Siegel.

In the fourth section we deal with the one dimensional case. Let K be a number field and $\sigma_0 : K \hookrightarrow \mathbb{C}$ is an embedding of K in $\mathbb{C}$. We also fix an embedding of the

algebraic closure $\overline{K}$ of K in $\mathbb{C}$. Let X be an N dimensional quasi projective variety defined over K; Let $S \subseteq X(\overline{K})$ and, for every positive integer r denote by S_r the set $\{x \in S \ s.t. \ [\mathbb{Q}(x) : K] \leq r\}$.

The main theorem of the fourth section is:

1.1 Theorem. *Let M be a parabolic Riemann surface (with a fixed positive singularity). Let $\gamma\colon M \to X(\mathbb{C})$ be an holomorphic map of finite order ρ with Zariski dense image. Suppose that, for every $\overline{K}$–rational point $P \in S \cap \gamma(M)$, the formal germ $\hat{M}_P$, of M near P, is (the pull back of) a LG– germ of type α (in its field of definition). then*

$$\frac{Card(\gamma^{-1}(S_r))}{r} \leq \frac{N+1}{N-1}\rho\alpha[K : \mathbb{Q}].$$

Since every algebraic Riemann surface is parabolic and the leaves of foliations (defined over K) are always LG–germs, this give a generalization of the Schneider–Lang Theorem:

1.2 Corollary. *Suppose we are in the hypotheses above, then*

$$\sum_{P \in \gamma^{-1}(S)} \frac{1}{[K(\gamma(P)); K]} \leq \frac{N+1}{N-1}\rho\alpha[K : \mathbb{Q}].$$

If X is a quasi projective variety defined over the number field K and r is a positive integer, denote by X_r the set $\{P \in X(\overline{K}) \ \text{ s.t. } \ [\mathbb{Q}(P) : K] \leq r\}$.

1.3 Corollary. *Let X be an algebraic variety defined over a number field K and let $F \hookrightarrow T_X$ be a foliation of rank one (defined over K). Suppose that the holomorphic foliation $F_\sigma \subset (T_X)_\sigma$ has a parabolic leaf M of finite order ρ (for some positive singularity on M) whose Zariski closure has dimension $d > 1$, then*

$$\frac{Card((X_r \setminus Sing(F)) \cap M)}{r} \leq \frac{d+1}{d-1}\rho[K : \mathbb{Q}].$$

Once one understand the one dimensional case, one realizes that the main tool used at infinite places is the *Evans Kernel*: cf. definition 4.6. This is a global solution of an elliptic differential equation and allows to bound the norm of the jet of a global section of a line bundle L on a point, in term of the growth of the first Chern form of L. This bound is exactly what is needed for arithmetic applications. Consequently, in the higher dimensional case one can develop a interesting (for arithmetic applications) value distribution theory once one suppose the existence of a function which generalize the properties of the Evans Kernel.

In the fifth section of the paper we define the notion of *conformally parabolic Khäler manifolds*. These are Khäler manifold where one can define an Evans kernel, cf. definition 5.1. One should notice the similarity between parabolic Riemann surfaces and

conformally parabolic varieties. We then show that the main example of conformally parabolic varieties is the complementary of a divisor in a compact Khäler manifold, thus in particular every quasi projective variety is conformally parabolic. We show that we can develop a value distribution theory for analytic maps from conformally parabolic varieties to projective varieties. Moreover, again the existence of the Evans kernel allows to bound the norm of the jet of a global section of a line bundle on a point, in terms of the growth of the first Chern class of the bundle. Thus we can develop the arithmetic consequences of this:

If X is a quasi projective variety of dimension N defined over the number field K and r is a positive integer, denote by X_r the set $\{P \in X(\overline{K}) \text{ s.t. } [\mathbb{Q}(P) : K] \leq r\}$. In section 5 we prove:

1.4 Theorem. *Let A be a d ($d < N$) dimensional conformally parabolic Khäler manifold with Khäler form ω. Let $\gamma\colon A \to X(\mathbb{C})$ be an analytic map of finite order ρ. Suppose that for every $p \in X_r \cap \gamma(A)$ the germ of $\gamma(A)$ near p is isomorphic to an LG–germ of type α. Suppose that the image of A is Zariski dense. Then, for every positive integer r, there exists a current T_r of bidegree $(1,1)$ on A with the following properties:*
– $\frac{1}{r} \cdot \int_A T_r \wedge \omega^{d-1} \leq \frac{N+1}{N-d}\rho\alpha[K : \mathbb{Q}]$;
– *For every $p \in \gamma^{-1}(X_r)$, the Lelong number $\nu(T_r, p)$ verifies $\nu(T_r, P) \geq 1$.*

In analogy with corollary 1.2, by taking weak limits, one obtain

1.5 Theorem. *Suppose that we are in the hypotheses as above. Then there exists a closed positive current T of bidegree $(1,1)$ over A such that:*
– $\int_A T \wedge \omega^{d-1} \leq 2\frac{N+1}{N-d}\rho\alpha[K : \mathbb{Q}]$;
– *for every $P \in \gamma^{-1}(X(\overline{K}))$ we have that $\nu(T; P) \geq \frac{1}{[K(\gamma(P)):K]}$.*

When A is an affine variety, there exists an analogue of Nevanlinna theory, developed for instance in [GK]. Thus there exists a notion of maps of finite order as soon as we fix an ample line bundle on a projective compactification $\overline{A}$ of A. This theory is very similar to the one dimensional case and it is more or less based on the theory of the Monge–Ampere differential equations. Unfortunately, at the moment it do not exists an analogue of the Evans kernel in the theory of the Monge–Ampere equations, consequently we do not know how to deal with the Cauchy inequalities over arbitrary affine variety when we are working with "classical" Nevanlinna theory. For this reason the value distribution theory we developped is more adapted to the arithmetic problems. In any case we prove that the order of growth in this theory is the same as the order computed in the classical theory of Griffiths and King.

As a consequence of 1.4 we find (with the same hypotheses as before on X)

1.7 Theorem. *Let A be an affine variety of dimension d ($d < N$) defined over $\mathbb{C}$ and $\gamma\colon A \to X(\mathbb{C})$ be an analytic map of finite order ρ. Suppose that for every $p \in S \cap \gamma(A)$*

the germ of $\gamma(A)$ near p is isomorphic to an LG–germ of type α. Suppose that the image of A is Zariski dense. Then for every positive integer r, the set $\gamma^{-1}(S_r)$ is contained in a hypersurface of degree at most $\frac{N+1}{N-d}\frac{d!\rho\alpha}{\deg(\overline{A})}[K:\mathbb{Q}]r$.

Once again we can deduce from this an explicit generalization of Bombieri–Schneider–Lang Theorem (cf. Corollary 5.34).

A posteriori one realizes that section 4 is a particular case of section 5 (thus one may wonder why we decided to write it). One should notice the following facts: section 4 lean on the preexhising theory of Parabolic Riemann surfaces and, as written it is completely self contained, thus a reader who is interested only on this easier case can skip the more complicated and involved section 5. Methods and definitions of section 5 are inspired by the corresponding of section 4; we think that the explicit proof in the one dimensional case allows to better understand and appreciate the higher dimensional situation. If one want to understand section 5 in the one dimensional case (and relate it with the classical theory of parabolic Riemann surfaces), one has to rewrite essentially all section 4! perhaps up to 4.15 and almost all what follows. For these reasons we decided to explicitely describe theory for parabolic Riemann surfaces before we develop the general case.

This paper owe a lot to J. B. Bost; the section on parabolic Riemann surfaces is due to him: he kindly explained it to me and sent the mail [Bo2] where the main ideas and techniques of §4 were explained. In §4 we reproduced his ideas. He also spent a lot of time explaining to me many techniques and ideas: also the application to the theory of vector bundles with integrable connections is suggested by him. I warmly thank him for all his help, patience and generosity. I thank the referee for the precious suggestions which highly improve the presentation of the paper: in particular the definition of conformally parabolic varieties have been suggested by him.

1.8 *Applications.*

We give an application of Theorem 3.8 based on the recent paper on classification of foliations on projective surfaces by M. McQuillan [MQ]: Let $FCP(\mathbb{C})$ be the category where the objects are couples (Y,f) where Y is a Riemann surface and $f\colon Y\to\mathbb{C}$ is a finite covering having ramification with polynomial grown: by this we mean that the function

$$N_f(r):=\sum_{0<|f(z)|<r} Ord_z(R_f)\log\left|\frac{r}{f(z)}\right|$$

where R_f is the ramification divisor, is a function of polynomial grown ($N_f(r):=O(r^\rho)$ for some ρ).

1.8 Theorem. *Let (X,F) be a foliated projective surface defined over a number field K (embedded in $\mathbb{C}$). Suppose that the foliation* do not *arises from a projective connection with irregular singular points and infinite monodromy. Suppose that Y is a*

curve in $FCP(\mathbb{C})$ contained in X and invariant for the foliation. Then, if $Y \cap X(K)$ is infinite, the Zariski closure of Y in X is a curve of genus less or equal then one.

Proof: We may suppose that the Zariski closure of Y is not an algebraic curve of genus less or equal then one. If the Zariski closure of Y is an algebraic curve of genus bigger or equal then two, by the classical Theorem of Faltings (Mordell conjecture) it has finitely many rational points. If the Zariski closure of Y is X, then, we may look to the classification of 2 dimensional foliations: in particular [MQ] chapter V and Fact V.1.2, the inclusion $Y \to X$ is an analytic map of finite order; so we may apply theorem 1.1 and conclude.

As an application of Theorem 1.7 to the theory of differential equations we can find the following; we suppose that K is a number field embedded in $\mathbb{C}$:

1.9 Theorem. *Let $\overline{X}_K$ be a smooth projective variety defined over K and D be a simple normal crossing divisor on it. Denote by X_K the quasi projective variety $\overline{X}_K \setminus D$. Let $(E;\nabla)$ be a vector bundle on $\overline{X}_K$ equipped with an integrable algebraic connection ∇ over X with meromorphic singularities along D. Let $f_{\mathbb{C}} : (X_K)_{\mathbb{C}}(\mathbb{C}) \to E(\mathbb{C})$ be an* analytic *horizontal section of $(E;\nabla)$ defined over $X_{\mathbb{C}} := X_K \times_K \mathbb{C}$. Let Y be the Zariski closure in X_K of the set $f^{-1}(E(K))$. Then the restriction of $f_{\mathbb{C}}$ to Y is an* algebraic *horizontal section of $(E,\nabla)|_Y$.*

Proof: We may suppose that X_K is smooth and affine. Taking as X_K a desingularization of an irreducible component of Y, we may suppose that $f^{-1}(E(K))$ is Zariski dense and eventually prove that $f_{\mathbb{C}}$ is algebraic. Remark that $f_{\mathbb{C}}^{-1}(E(K)) \subseteq X_K(K)$, consequently to require that $f_{\mathbb{C}}^{-1}(E(K))$ is Zariski dense on X_K is equivalent to require that it is Zariski dense on $X_K \times_K \mathbb{C}$. We claim that the section $f_{\mathbb{C}}$ is of finite order:

We may work in a neighborhood U of a point of the divisor D. We may suppose that U is a polydisk with coordinates $z_1, \dots, z_n$ and that D is given by $z_1 \cdot z_2 \cdot \dots \cdot z_r = 0$.

The connection is given by the linear differential equation

$$dy = \frac{A}{z_1^{i_1} \cdot \ldots z_r^{i_r}} y$$

With A a matrix of holomorphic forms.

Let $z = (z_1, \dots, z_n)$ with $|z_i| = 1$ and $t \in [0,1]$ and consider the function $h(t) := |y(tz)|$. By Lipschitz condition, there is a suitable constant C such that

$$|h'(t)| \leq |z \cdot dy(tz)| = |dy(tz)| \leq \frac{C}{t^{i_1+\cdots+i_r}} h(t).$$

Observe that $h(t)$ is decreasing, thus, $h'(t) < 0$. Diving by $h(t)$ and integrating both sides we obtain

$$\log(h(t)) \leq \frac{C'}{t^a}$$

for suitable C' and a. The claim follows.

We apply now Theorem 1.7 to X_K and we obtain that, if $f_{\mathbb{C}}$ is not algebraic, $f^{-1}(E(K))$ cannot be Zariski dense. From this we conclude.

1.10 Remark. One should notice that, an analytic function, say over $\mathbb{C}^d$, is of finite order ρ if there exists a constant $C > 0$ such that $\sup_{\|z\|\leq R}\{|f(z)|\} \leq C^{R^\rho}$. This definition is not the same as the definition given in [Dl].

2 The general algebraization setting.

Let K be a number field, O_K be its ring of integers, S_∞ the set of infinite places, M_{fin} the set of finite places of K and $M_K = S_\infty \cup M_{fin}$. We fix once for all an embedding $\sigma_0 : K \hookrightarrow \mathbb{C}$ (so an infinite place of K). Let X_K be a geometrically irreducible, connected quasi projective variety of dimension N defined over K and $F \subseteq X(K)$ a finite subset. For $P \in F$ we denote by $\widehat{X}_P$ the completion of X_K at P. for every $P \in F$, let $\widehat{V}_P \subseteq \widehat{X}_P$ be a smooth formal subvariety of dimension d. For every (finite or infinite) place $\mathfrak{p}$ of K, we suppose that the base change $V_P^{\mathfrak{p}}$ is analytic: the power series defining it have radius of convergence which is non zero.

2.1 Definition. *Let $P \in X(K)$ and $\widehat{V}_P \subseteq \widehat{X}_P$ be a formal subscheme. We will denote by $\bar{V}_P$ the smallest* Zariski *closed set, defined over K, containing $\widehat{V}_P$ and call it* the Zariski closure of $\widehat{V}_P$.

We will say that $\widehat{V}_P$ is algebraic *if* $\dim \widehat{V}_P = \dim \bar{V}_P$.

In this chapter we would like to give some general sufficient condition in order to have that each of the $\widehat{V}_P$'s is algebraic.

By replacing X_K with the Zariski closure of the $\widehat{V}_P$'s we may suppose that at least one of the $\widehat{V}_P$'s is Zariski dense.

We fix a model $\mathcal{X}$ of X_K flat, projective over $\mathrm{Spec}(O_K)$. Moreover for every infinite place σ we fix a Kähler metric on X_σ.

Let L be a relatively ample line bundle on $\mathcal{X}$ and, we suppose that, for every infinite place σ the holomorphic line bundle L_σ is equipped with a smooth positive metric.

Consequently, for every positive integer D, the locally free O_K–module $E_D := H^0(\mathcal{X}; L^D)$ is equipped with the L^2 and the sup norms which are comparable by, for instance [Bo]; so E_D has the structure of an hermitian vector bundle over O_K.

For every $P \in F$ we fix an hermitian integral structure on the K–vector space $T\widehat{V}_P$. Consequently, for every place $\mathfrak{p}$ of K, and for every $(i, D) \in \mathbb{N} \times \mathbb{N}$, the vector space $\left(S^i(T_P^*\widehat{V}) \otimes L^D|_P)\right)_{\mathfrak{p}}$ is equipped with a $\mathfrak{p}$–adic norm (which, at infinite places is a metric).

Let $(V_P)_i$ the i–th infinitesimal neighborhood of P in $\widehat{V}_P$; there is a canonical exact sequence

$$0 \longrightarrow S^i(T_P^*\widehat{V}) \otimes L^D|_P) \longrightarrow L^D|_{(V_P)_{i+1}} \longrightarrow L^D|_{(V_P)_i}.$$

For every $(i;D) \in \mathbb{N} \times \mathbb{N}$, we denote by E_D^i the kernel of the map α_D^i given by the composite

$$E_D \hookrightarrow \bigoplus_{P_j \in F} H^0(\widehat{V}_{P_j}; L^D) \longrightarrow \bigoplus_{P_j \in F} H^0((\widehat{V}_{P_j})_i; L^D) \tag{2.1.1}$$

remark that the first map is injective because the germs are Zariski dense. We denote by γ_D^i the induced map

$$\gamma_D^i : E_D^i / E_D^{i+1} \hookrightarrow \bigoplus_{P_j \in F} H^0(P_j; S^i((T_{P_j}\widehat{V}_{P_j})^*) \otimes L^D|_{P_j}). \tag{2.2.1}$$

For every (finite or infinite) place $\mathfrak{p}$, the $K_\mathfrak{p}$–vector space $(E_D^i)_\mathfrak{p}$ is equipped with the norm induced by the norm of $(E_D)_\mathfrak{p}$. Consequently the $K_\mathfrak{p}$–vector space $\left(E_D^i/E_D^{i+1}\right)_\mathfrak{p}$ is equipped with the quotient norm. We will denote by $\|\gamma_D^i\|_\mathfrak{p}$ the $\mathfrak{p}$–norm of the linear map γ_D^i.

An easy computation show that $\frac{1}{[K:\mathbb{Q}]}\sum_{\sigma \in M_K} \log \|\gamma\|_\sigma$ do not depend on the field K.

2.3 Theorem. *With the notation as above, suppose that the formal germs are* not algebraic *and that we can choose the hermitian integral structure on the tangent spaces $T_P\widehat{V}$'s in such a way that the following holds: there exist constants C_1, C_2, λ and A depending only on F, the model $\mathcal{X}$ and the choice of the hermitian integral structure, but independent on the i and D, for which:*

1) the following inequality holds:

$$\frac{1}{[K:\mathbb{Q}]} \sum_{\mathfrak{p} \in M_{fin}} \log \|\gamma_D^i\|_\mathfrak{p} \leq C_1(i \log i + C_2(i+D));$$

2) the following inequality holds

$$\frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in S_\infty} \log \|\gamma_D^i\|_\sigma \leq C_2(i+D);$$

3) for $\frac{i}{D} \geq \lambda$, the following inequality holds

$$\frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in S_\infty} \log \|\gamma_D^i\|_\sigma \leq C_1 \left(-Ai \log\left(\frac{i}{D}\right) + C_2(i+D) \right).$$

Then $A \leq \frac{N+1}{N-d}$ (recall that $N = dim(X)$ and $d = \dim(V_P)$).

At first glance theorem 2.3 may seems strange. One should understand it in this way: given some formal germs, if one can prove that the norms of the involved maps are *very*

small then they are algebraic. One will see in the other sections that the bounds of the norms is related to the arithmetic and the geometry of the formal germs.

Before we start the proof, we recall the main slope inequalities, proved, for instance in [Bo], §4.1 (we refer to loc. cit. for the notation):

a) If E is an hermitian vector bundle over O_K, then we call the real number $\mu_n(E) := \frac{1}{[K;\mathbb{Q}]} \cdot \frac{\widehat{\deg}(E)}{rk(E)}$, *the slope* of E;

b) within all the sub bundles of a given hermitian vector bundle E, there is one having maximal slope; we call its slope *the maximal slope of* E and denote it by $\mu_{\max}(E)$;

c) If E is an hermitian vector bundle and $S^i(E)$ its i–th symmetric power metric. we endow $S^i(E)$ with the symmetric power metric of E. We can find a constant (depending on E) $B \in \mathbb{R}$, such that $\mu_{\max}(S^i(E)) \leq iB$;

d) if E_1 and E_2 are two hermitian vector bundles, then

$$\mu_{\max}(E_1 \oplus E_2) = \max\{\mu_{\max}(E_1); \mu_{\max}(E_2)\};$$

e) (the main slope inequality): Suppose that F_K is a finite dimensional vector space endowed with a filtration $\{0 = F^{r+1}\} \subset F^r \subset F^{r-1} \subset \ldots \subset F^0 = F_K$ such that the successive sub quotients $G^i_K = F^i/F^{i+1}$ are the generic fibre of hermitian vector bundles G^i. Suppose moreover that we have an hermitian vector bundle E and an *injective* linear map $\varphi_K : E_K \hookrightarrow F_K$. Denote by $E^i_K := \varphi^{-1}(F_i)$ and by $E^i := E^i_K \cap E$. For every i, we have an induced map $\varphi_i : E^i \to G^i$. For every (finite or infinite) place $\mathfrak{p}$, we denote by $\|\varphi_i\|_{\mathfrak{p}}$ the $\mathfrak{p}$–adic norm of the linear map φ_i; then

$$\frac{\widehat{\deg}(E)}{[K;\mathbb{Q}]} \leq \sum_{i=0}^{r} rk\left(E^i/E^{i+1}\right)\left[\mu_{\max}(G^i) + \sum_{\mathfrak{p}} \log \|\varphi_i\|_{\mathfrak{p}}\right];$$

f) (Arithmetic Hilbert–Samuel formula) There exists a positive constant C such that

$$\frac{\widehat{\deg}(E_D)}{[K;\mathbb{Q}]} \geq -CD^{N+1}$$

(where we recall that $N = \dim X_K$).

We can now prove theorem 2.3:

Proof: (of Theorem 2.3) We suppose that at least one of the $\widehat{V}_P$'s is *not* algebraic and that $A > \frac{N+1}{N-d}$ and we eventually find a contradiction.

Choose an $\epsilon > 0$, such that $\alpha := \frac{N+1}{d+1} - \epsilon > 1$ and $\alpha > \frac{A}{A-1}$.

Take D such that $D^\alpha > \lambda D$. Then, by the main slope inequality applied to the map 2.1.1 with the filtration induced by 2.2.1 and the arithmetic Hilbert–Samuel formula, we can find constants C_j's, independent on i and D, (provide that they are sufficiently

big) such that

$$
\begin{aligned}
-C_0 D^{N+1} &\leq \sum_{i=0}^{\infty} rk\left(E_D^i/E_D^{i+1}\right)\left[C_2(i+D)+i\log i+\sum_{\sigma\in S_\infty}\log\|\gamma_D^i\|_\sigma\right]\\
&\leq \sum_{i\leq D^\alpha} rk\left(E_D^i/E_D^{i+1}\right)\left[C_3D^\alpha+C_4D^\alpha\log D\right]\\
&+\sum_{i>D^\alpha} rk\left(E_D^i/E_D^{i+1}\right)\left[C_5\left(-Ai\log\left(\frac{i}{D}\right)+i\log i\right)+C_2(i+D)\right].
\end{aligned}
$$

Consequently

$$
\begin{aligned}
C_0 \geq\ & \frac{rk\left(E_D/E_D^{D^\alpha}\right)\left[-C_3D^\alpha-C_4D^\alpha\log D\right]}{D^{N+1}}\\
&+\sum_{i>D^\alpha}\frac{rk\left(E_D^i/E_D^{i+1}\right)}{D^N}\left(C_5\frac{i}{D}\left[(A-1)\log i-A\log D\right]-C_2\left(\frac{i}{D}+1\right)\right).
\end{aligned}
$$

If we denote by I_j the ideal of the point P_j on the local germ $\widehat{V}_{P_j}$, we have an injection

$$
E_D/E_D^{D^\alpha}\hookrightarrow\bigoplus_{P_j\in F}L^D|_{P_j}\otimes\mathcal{O}_{\widehat{V}_{P_j}}/I_j^{D^\alpha};
$$

thus $rk(E_D/E_D^{D^\alpha})=O(D^{d\alpha})$. This, together with our choice of α, implies that

$$
\lim_{D\to\infty}\frac{rk(E_D/E_D^{D^\alpha})\left(-C_3D^\alpha-C_4D^\alpha\log D\right)}{D^{N+1}}=0.
$$

On the other side, for $i\geq D^\alpha$, we have

$$
(A-1)\log i\geq\alpha(A-1)\log D;
$$

consequently, again because of our choice of α, we can find a positive constant β such that

$$
(A-1)\log i-A\log D>\beta\log D.
$$

Thus

$$
\begin{aligned}
&\sum_{i>D^\alpha}\frac{rk\left(E_D^i/E_D^{i+1}\right)}{D^N}\left(C_5\frac{i}{D}\left[(A-1)\log i-A\log D\right]-C_2\left(\frac{i}{D}+1\right)\right)\\
&\sum_{i>D^\alpha}\frac{rk\left(E_D^i/E_D^{i+1}\right)}{D^N}\left[C_5\frac{i}{D}\beta\log D-C_2\left(\frac{i}{D}+1\right)\right]\\
&\geq\sum_{i>D^\alpha}\frac{rk\left(E_D^i/E_D^{i+1}\right)}{D^N}\left[C_5\frac{i}{D}\left(\beta\log D-C_6\right)-C_2\right].
\end{aligned}
$$

If we choose D so big such that $\beta\log D-C_6\geq0$ we obtain that the last displayed

sum is greater than

$$C_6 \sum_{i>D^\alpha} \frac{rk\left(E_D^i/E_D^{i+1}\right)}{D^N}\left[D^{\alpha-1}-1\right]$$
$$= C_6 \frac{rk(E_D^{D^\alpha})}{D^N}\left(D^{\alpha-1}-1\right);$$

but since $\lim_{D\to\infty} \frac{rk(E_D^{D^\alpha})}{D^N} = 1$, we eventually find a contradiction as soon as D is sufficiently big.

Once we assure the conditions (1), (2) (3), Theorem 2.3 gives us a powerful tool for algebraization of analytic germs of subvarieties of a projective variety. We remark that the classical Cauchy inequality assures that, if the formal germ has a positive radius of convergence at every infinite place, condition (2) is automatic. On the other side conditions (1) and (3) not always hold. If one chose the integral structures involved in a an arbitrary way there is no way a priori to give explicit bounds as required by the hypotheses of the theorem. One may think the problem in this way: the choices of the metrics at infinity may be done by some analytic information of the germs; the choice of the integral structure is related to p–adic information of the germs and the choice of the constant A is purely arithmetic.

3 *LG*–germs.

We describe in this chapter a class of germs of analytic subvarieties of a variety defined over a number field which verify condition (1) of Theorem 2.3. Roughly speaking germs in this class are defined by power series whose coefficients of each monomial are algebraic integers divided by the factorial of its exponents. the example to keep in mind is the classical exponential function. The other leading example is the formal leaf of an analytic foliation in a smooth point of the foliation.

At the moment we do not know the right condition a germ should have in order to apply theorem 2.3 (beside the fact that it must verify (1) of loc. cit.). The referee proposed (and we warmly thank her/him for this) very interesting remarks and observations on the Gevrey condition on analytic power series. We hope to come again on this in a future paper.

3.1 *Good germs.* Let K be our number field and $S \in M_K$ be a finite set of places of K (containing all the infinite places); let O_S be the ring of S–integers of K:

$$O_S := \{a \in K/\ \|a\|_v \leq 1 \,\text{for every} v \notin S\}.$$

We will put $B := Spec(O_S)$. In general if R is scheme, we will denote by $\mathbb{A}^N_R$ the N–dimensional affine space over R and by $\widehat{\mathbb{A}}^N_R$ the completion of it at the origin 0.

Let $f : \mathcal{X} \to B$ be an irreducible scheme, flat over B and $\mathcal{P} : B \to \mathcal{X}$ be a rational point. We will suppose that f is smooth in a neighborhood of $\mathcal{P}(B)$. When $f_K : \mathcal{X}_K \to Spec(K)$ is smooth around $\mathcal{P}_K$, this will be the case up to enlarge S, if necessary.

Let $N = \dim(\mathcal{X}_K)$.

Let $\widehat{\mathcal{X}_P}$ be the completion of $\mathcal{X}_K$ around $P := \mathcal{P}_K$.

Since $\mathcal{X}$ is smooth over B around $\mathcal{P}$, we can choose an open neighborhood $U \subset \mathcal{X}$ of P and an étale map $g : U \to \mathbb{A}^N_B$. We can (and we will) suppose that $g(\mathcal{P}) = 0$.

By definition of étale map, the completion of g will induce an isomorphism $\hat{g} : \widehat{\mathcal{X}_P} \xrightarrow{\sim} \widehat{\mathbb{A}^N_B}$.

We fix coordinates $\mathbf{Z} = (Z_1; \ldots; Z_N)$ on $\widehat{\mathbb{A}}^N_B$ and $\mathbf{T} := (T_1; \ldots; T_d)$ on $\widehat{\mathbb{A}}^d_B$. This choice induces coordinates on $\widehat{\mathbb{A}}^N_K$ and on $\widehat{\mathbb{A}}^d_K$.

Let $\gamma : \widehat{\mathbb{A}}^d_K \to \widehat{\mathcal{X}_P}_K$ be a formal morphism (such that $\gamma(0) = P$). The map $h := \hat{g} \circ \gamma \colon \widehat{\mathbb{A}}^d_K \to \widehat{\mathbb{A}}^N_K$ is given by N power series

$$h(\mathbf{T}) = \mathbf{Z}(\mathbf{T}) = (Z_1(\mathbf{T}); \ldots; Z_N(\mathbf{T}))$$

with $Z_i(\mathbf{T}) := \sum_{I \in \mathbb{N}^d} a_I(i) \mathbf{T}^I$ and $\mathbf{Z}(0) = 0$. Where, here and in the sequel, if $I = (i_1, \cdot, i_d) \in \mathbb{N}^d$, we will denote $\mathbf{T}^I := T_i^{i_1} T_2^{i_2} \cdots T_d^{i_d}$ moreover we will write $I! := i_1! \cdot i_2! \cdot \ldots \cdot i_d!$ and $|I| := i_1 + \cdot + i_d$.

3.1 Definition. *Let α be a real number. The map γ will be said to be a S*–Good germ of type α *with respect to $\mathcal{X}$ (or simply a S–Good germ, if α is clear from the contest), if:*

a) For every $\mathfrak{p} \notin S$, there in a constant $C_\mathfrak{p} \geq 1$ such that

$$\|a_I(i)\|_\mathfrak{p} \leq \frac{C_\mathfrak{p}^{|I|}}{\|I!\|_\mathfrak{p}^\alpha} \quad \text{and} \quad \prod_{\mathfrak{p} \notin S} C_\mathfrak{p} < \infty.$$

b) The $a_I(i)$ with $|I| = 1$ are in O_S and the linear map $\alpha : O_S^d \to O_S^N$ given by the matrix $(a_I(i))$ with $|I| = 1$ is is injective and has cokernel without torsion.

The second condition can be geometrically explained in the following way: we have an induced linear map of tangent spaces

$$d\gamma : T_0(\widehat{\mathbb{A}^d_B})) \longrightarrow T_P \mathcal{X}_K$$

in both tangent spaces there is a natural O_S module of maximal rank given by the tangent space of their integral models. Condition b) is equivalent to ask that the map $d\gamma$ is the extension of a linear map between these O_S–modules which is injective modulo every prime ideal of O_S. We observe that a S–good germ is a smooth formal subscheme of $\widehat{\mathcal{X}_{P_K}}$.

In the sequel we will use the following lemma

3.2 Lemma. *For every positive integer i, the following holds*

$$-\sum_{\mathfrak{p}\in M_{fin}} \log\|i!\|_{\mathfrak{p}} \leq i\log(i) - i + O(1).$$

The proof is an easy consequence of the product formula and the Stirling's formula (we leave details to the reader).

The power series $Z_i(\mathbf{T})$ define a S–Good germ if and only, for every $\mathfrak{p}$ there is a constant $C_{\mathfrak{p}}$ with the same properties as definition 3.1, such that, for every multiindex I (with the evident notation)

$$\|\frac{\partial^I Z_i(T)}{\partial T^I}(0)\|_{\mathfrak{p}} \leq \frac{C_{\mathfrak{p}}^{|I|}}{\|I!\|_{\mathfrak{p}}^{\alpha-1}}.$$

A useful property of S–Good germs is deduced from the following lemma, the proof of which is done by easy induction:

3.3 Lemma. *Let $W_i := \sum_I a(i)T^I \in K[\![\mathbf{T}]\!]$ $(i = 1, 2)$ be two power series. Suppose that there are constants C_i such that*

$$\|a_I(i)\|_{\mathfrak{p}} \leq \frac{C_i^{|I|}}{\|I!\|_{\mathfrak{p}}^{\alpha}}.$$

Then $W_1 \cdot W_2 = \sum_I b_I T^I$ with $\|b_I\|_{\mathfrak{p}} \leq \frac{(C_1\cdot C_2)^{|I|}}{\|I!\|_{\mathfrak{p}}^{\alpha}}$.

The property of being S– good, for a germ γ, seems to depend on the choice of the coordinates on $\widehat{\mathbb{A}}_S^d$, on $\widehat{\mathbb{A}}_S^N$ and on the map g; so it would be more precise to speak about "S–goodness with respect to $(\mathbf{T}, g, \mathbf{Z})$". As the lemmas below will show, the fact that a map γ is a S–good germ depends only on γ itself and on the model $\mathcal{X}$. So we can speak about S– good germs over a flat S–scheme without making reference on any choice.

3.4 Lemma. *Let*

$$Aut_S(\widehat{\mathbb{A}}_B^d) := \{F(\mathbf{T}) = (f_1(\mathbf{T}); \ldots; f_d(\mathbf{T}))\ s.t.$$
$$f_i(\mathbf{T}) = \sum_{J\in\mathbb{N}^d} b_J T^J,\ b_J \in O_S \text{ and } \det(\frac{\partial f_i}{\partial t_j})(0) \in O_S^*\}$$

be the group of S–automorphisms of $\widehat{\mathbb{A}_B^d}$. If $\mathbf{T} = F(\mathbf{T}')$ for some $F \in Aut_S(\widehat{\mathbb{A}_S^d})$, then the germ γ is S–good with respect to $(\mathbf{T}; g; \mathbf{Z})$ if and only if it is S–good with respect to $(\mathbf{T}'; g; \mathbf{Z})$.

The definition of $Aut_S(\widehat{\mathbb{A}_S^d})$ is intrinsic it do not depend on the choice of the coordinates: for every finite place $\mathfrak{p}$ of O_S denoting by $\Delta_{\mathfrak{p}}^d$ the poly–disk $\{(x_1; \ldots; x_d) \in$

$K_{\mathfrak{p}}^d \,/\, \|x_i\|_{\mathfrak{p}} < 1\}$; $Aut_S(\widehat{\mathbb{A}^d_S})$) the is the group of the formal automorphisms F of $\widehat{\mathbb{A}^d_K}$ such that, for every $\mathfrak{p} \notin S$ the restriction of F to $\widehat{\mathbb{A}}^d_{K_{\mathfrak{p}}}$ converges on $\Delta^d_{\mathfrak{p}}$ and $F(\Delta^d_{\mathfrak{p}}) = \Delta^d_{\mathfrak{p}}$.
Proof: With the notations as in 3.1, given a good germ γ and an automorphism $F \in Aut_S(\widehat{\mathbb{A}}^d_S)$ we have a composite of maps

$$\widehat{\mathbb{A}^d_S} \xrightarrow{F} \widehat{\mathbb{A}^d_S} \xrightarrow{h} \widehat{\mathbb{A}^N_S}$$

and the corresponding map between tangent spaces. Since $F \in Aut_S(\widehat{\mathbb{A}^d_S})$, the linear map dF is an isomorphism of the integral models, so condition b) holds for T' (and for g and $\mathbf{Z}$).

We show now that, if condition (a) holds for $(\mathbf{T}; g; \mathbf{Z})$ it holds for $(\mathbf{T}'; g; \mathbf{Z})$. Let $\mathbf{Z}_i(\mathbf{T}') = \mathbf{Z}_i(F(\mathbf{T}'))$ we must show that, there exists constants $C_{\mathfrak{p}}$ (with the condition on all the $C_{\mathfrak{p}}$'s) and α such that, for every $I = (i_1; \ldots, i_d) \in \mathbb{N}^d$,

$$\|\frac{\partial^I}{(\partial^I T')}\mathbf{Z}(0)\|_{\mathfrak{p}} \le \frac{C_{\mathfrak{p}}^{|I|}}{\|I!\|_{\mathfrak{p}}^{\alpha}}.$$

by induction on the multi–index I (ordered by lexicographic order) it is easy to see that for a given $I \in \mathbb{N}^d$ there exists a polynomial $P(X_\alpha; Y^h_\beta) \in \mathbb{Z}[X_\alpha; Y^h_\beta]$ where $\alpha \in \mathbb{N}^d$ is such that $\alpha \le I$ and β and h are suitable multi–index having the following properties:
i) $\deg P(X_\alpha; Y^h_\beta)_{X_\alpha} \le 1$;
ii) $\frac{\partial^I}{(\partial^I T')}\mathbf{Z}(\mathbf{T}') = P(\frac{\partial^\alpha \mathbf{Z}(F(\mathbf{T}'))}{(\partial \mathbf{T})^\alpha}; \frac{\partial^h F_\beta(\mathbf{T}')}{(\partial \mathbf{T}')^h})$.
If we put $\mathbf{T}' = 0$ and use the hypotheses we easily conclude.

3.5 Remark. Observe that, the constants $C_{\mathfrak{p}}$ may depend on the parametrization but the constant α depends only on the germ.

Using the same strategy we prove that the notion of S–Good germ independent on the choice of the coordinates on $\widehat{\mathbb{A}}^N_S$:

3.6 Lemma. *If $\mathbf{W} = G(\mathbf{Z})$ for some $G \in Aut_S(\widehat{A^N_S})$ then the germ γ is S–Good with respect to $(\mathbf{T}; g; \mathbf{W})$ if and only if it is S–Good with respect to $(\mathbf{T}; g; \mathbf{Z})$.*

The fact that the notion of S–good germ is independent on the choice of g follows also from Lemma 3.6.

3.7 *Definition of LG–germ.*

Let X_K be a smooth variety defined over K of dimension N and $p \in X_K(K)$. Let $\widehat{X_P}$ be the formal neighborhood of X_K around P and $\iota : \widehat{V} \hookrightarrow \widehat{X_P}$ be a smooth formal subscheme of dimension d.

3.7 Definition. *Let α be a real number. The formal scheme $\widehat{V}$ is a LG–*germ of type *α (or simply a LG–germ) if the following holds:*

i) For all $\mathfrak{p} \in \operatorname{Spec}\max(O_K)$ the induced formal subscheme $\widehat{V}_{K_\mathfrak{p}} \hookrightarrow (\widehat{X_P})_{K_\mathfrak{p}}$ is an analytic germ: the equations defining it have a positive radius of convergence.

ii) There exists a finite set S of places of K (containing all the infinite places) with:

– a smooth model $\mathcal{X}_S \to Spec(O_S)$ of X_K over which the rational point extends to a section $P : Spec(O_S) \to \mathcal{X}_S$,

– a S–good germ of type α with respect to $\mathcal{X}$, $\gamma : \widehat{\mathbb{A}}^d_K \hookrightarrow \widehat{\mathcal{X}_{S\,P}}$

– a K–isomorphism $\delta : \widehat{\mathbb{A}}^d_K \xrightarrow{\sim} \widehat{V}$ such that $\gamma = \iota \circ \delta$.

3.8 Remark. Suppose that $\iota : \widehat{V} \to (\widehat{X_K})_P$ is an LG–germ and $\mathcal{X} \to Spec(O_K)$ is a model of X_K. then we can find a finite set of places S a model $\mathcal{X}'$ of X_K, a birational map $\mathcal{X}' \to \mathcal{X}$ and a S–good germ with respect to $\mathcal{X}'$, $\gamma : \widehat{\mathbb{A}}^d_K \hookrightarrow \widehat{X_P}$ with a K–isomorphism $\delta : \widehat{\mathbb{A}}^d_K \xrightarrow{\sim} \widehat{V}$ such that $\gamma = \iota \circ \delta$. consequently, the notion of LG–germ is essentially independent on the choice of the model.

One of the leading example of LG-germ is the formal leaf of a foliation in a smooth point:

3.9 Proposition. *X_K be a smooth variety over K and $\mathcal{F} \hookrightarrow T_{X_K}$ be a foliation on it. Let $P \in X_K(K)$ be a smooth point for the foliation and $V_P \hookrightarrow \widehat{X_{K\,P}}$ the formal leaf of the foliation through P. Then V_P is a LG–germ of type 1.*

3.10 Proposition. . *Suppose we are in the hypotheses of 3.9 and moreover for almost every $\mathfrak{p}$ the foliation is closed under the p–derivation; then the formal leaf of the foliation through a smooth point is a LG–germ of type zero.*

The proofs are in [Bo] page 189 and ff.

Let X_K, $P \in X_K(K)$ and $\widehat{V} \hookrightarrow \widehat{X_P}$ be a smooth formal germ. An integral model of X_K give rise to an integral structure on the tangent space of $\widehat{V}$:

3.11 *Construction:*. Let $f : \mathcal{X} \to \operatorname{Spec}(O_K)$ be a model of X_K where P extends to a section $\mathcal{P} : Spec(O_K) \to \mathcal{X}$. Denote by $T^*(\cdot)$ the cotangent space. By definition we have a surjective map $(d\iota)^* : T^*_P X_K \twoheadrightarrow T^*_P\widehat{V}$ and $T^*_P X_K$ is the generic fibre of the O_K–module $T^*_{\mathcal{P}}\mathcal{X}$; the O_K– module $(d\iota)^*(T^*_{\mathcal{P}}\mathcal{X})$ is then an integral model of $T^*_P\widehat{V}$. We will denote it by $\mathcal{T}^*_P\widehat{V}$.

Let X_K and $P \in X_K(K)$ as above. Let $\widehat{V} \hookrightarrow \widehat{X_P}$ be a LG–germ. Denote by $\widehat{V}_i$ the i–th infinitesimal neighborhood of P in $\widehat{V}$. There is a natural exact sequence

$$0 \longrightarrow S^i(T^*_P\widehat{V}) \longrightarrow \mathcal{O}_{\widehat{V}_{i+1}} \longrightarrow \mathcal{O}_{\widehat{V}_i} \longrightarrow 0.$$

Let $U \subset X_K$ be a Zariski open neighborhood of P and $s \in H^0(U, \mathcal{O}_{X_K})$. Suppose that the restriction of s to $\widehat{V}_{i-1}$ is zero (we will say that s vanishes at order $i-1$ along $\widehat{V}$). Then, the restriction of s to $\widehat{V}_i$ canonically defines a section in $S^i(T^*_P\widehat{V})$. We will denote this section by $j^i(s)$ and call it the *i–th jet of s.*

Let $\mathcal{X} \to \operatorname{Spec}(O_K)$ be a model of X_K and suppose that P extends to a section $P\colon \operatorname{Spec}(O_K) \to \mathcal{X}$.

The interest of LG–germs is that, we can actually bound from above the norm of the i–th jet of integral functions.

3.12 Theorem. *Let $\widehat{V} \hookrightarrow (\widehat{X_K})_P$ be a LG–germ of type α and $\mathcal{X} \to \operatorname{Spec}(O_K)$ be a model of X_K. Then we can find a constant C for which the following holds:*

Let $U \subseteq \mathcal{X}$ be a Zariski open neighborhood of P and $s \in H^0(U, \mathcal{O}_{\mathcal{X}})$. Suppose that s vanishes at the order $i-1$ along $\widehat{V}$ then

$$\sum_{\mathfrak{p} \in \operatorname{Spec}\max(O_K)} \log \|j^i(s)\|_{\mathfrak{p}} \leq \alpha i \log(i) + Ci.$$

Proof: First of all, we remark that, since $\widehat{V}$ is a LG–germ we can work with the involved map $\gamma\colon \widehat{\mathbb{A}}^d_K \to (\widehat{X_K})_P$. We fix the set S and we can suppose that the smooth model $\mathcal{X}' \to B := \operatorname{Spec}(O_S)$ involved in the definition of the LG–germ is $\mathcal{X}|_B$. We fix an open neighborhood $U \subseteq \mathcal{X}|U$ of P with an étale map $g\colon U \to \mathbb{A}^N_B$. Fix coordinates $\mathbf{Z} := (Z_1, \ldots, Z_N)$ on $\widehat{\mathbb{A}}^N_B$ and coordinates $\mathbf{T} := (t_1, \ldots, t_d)$ on $\widehat{\mathbb{A}}^d_B$. The restriction to B of the integral structure of $S^i(T^*_P\widehat{V})$ is the O_S–module $\mathcal{S}^i := \bigoplus_{i_1+\ldots+i_d=i} O_S(dt_1)^{i_1}\cdots(dt_d)^{i_d}$.

Moreover the restriction of s to $\widehat{\mathbb{A}}^N_B$ is a power series $s = \sum_{I\in\mathbb{N}^N} a_I Z^I$ with $a_I \in O_S$. The restriction via γ of s to $\widehat{\mathbb{A}}^d_K$ is then the power series $s|_{\widehat{\mathbb{A}}^d_K} = \sum a_I \mathbf{Z}^I(\mathbf{T})$. Thus, considering Taylor extension,

$$j^i(s) = \sum_{i_1+\ldots+i_d=i} \frac{1}{i_1!\cdot i_2!\cdot\ldots\cdot i_d!} \cdot \frac{\partial^i s(\mathbf{T})}{\partial^{i_1}t_i \ldots \partial^{i_d}t_d}|_{\mathbf{T}=(0)}(dt_1)^{i_1}\cdots(dt_d)^{i_d}.$$

The conclusion follows by induction on I and 3.3 applied to Z^I.

As corollary we find the estimation at finite places needed to apply Theorem 2.3. Let X_K be a smooth projective variety defined over K, $P \in X_K(K)$ a K–rational point and $\iota : \widehat{V} \to X_K$ be a LG–germ. Suppose that L is an ample line bundle on $\mathcal{X}$ (a suitable model of X_K) equipped with a positive hermitian metric. We suppose that we fixed a positive metric on $X_K(\mathbb{C})$. Let γ^i_D the linear maps defined in 2.2.1; then

3.13 Corollary. *Let X_K be a smooth variety defined over K, $P \in X_K(K)$ and L as above. Let $\widehat{V} \subseteq \widehat{X_P}$ be a smooth formal subvariety. Suppose that $\widehat{V}$ is an LG–germ of type α. Then for every model $\mathcal{X}$ of X_K over O_K we can choose an integral model of the tangent space of $\widehat{V}$ at P and a positive constant C depending only on the model, on the point P and on $\widehat{V}$ such that*

$$\sum_{\mathfrak{p}\in M_{fin}} \log \|\gamma^i_D\|_{\mathfrak{p}} \leq [K:Q]\left(\alpha \cdot i \log i + Ci\right).$$

In order to prove this corollary we use Theorem 3.12 to estimate the norms at the places of O_S and the standard (p–adic) Cauchy inequality for the the finite set of places S.

4 Rational points on parabolic Riemann surfaces.

Let X_K be a projective variety of dimension $N > 1$ defined over some number field K and $S \subset X_K(K)$. For every point $P \in S$, we fix a LG–germ $\widehat{V}_P$ of dimension one on P. Suppose that, there is a Riemann surface M and an analytic map $\gamma : M \to X_K(\mathbb{C})$ (where we see $\mathbb{C}$ as a K–algebra via the embedding σ_0 fixed in §1), passing through S. Let $F \in M$ be a subset such that $\gamma(F) \subset S$. Suppose that, for every $Q \in F$, the germ of curve defined by $\gamma(M)$ near $\gamma(Q)$ coincides with $\widehat{V}_{\gamma(Q)}$. In this chapter we will suppose that M is parabolic (cf.below) and the map γ is of finite order ρ with respect to some positive singularity τ on M. The aim of this chapter is to show that, if the cardinality of F is very big with respect to ρ, then the image of γ is algebraic. The theorem we will prove is more general: it will concern points defined over extensions of bounded degree.

4.1 *Parabolic Riemann surfaces.*

4.1 Definition. *A Riemann surface M is said to be parabolic if every upper bounded subharmonic function on it is constant.*

4.2 *Examples.* a) The complex plane $\mathbb{C}$ is parabolic;

b) every algebraic Riemann surface is parabolic (affine or projective);

c) if F is a set of capacity zero on a parabolic Riemann surface M then $M \setminus F$ is again parabolic: so, for instance, the complementary set of a lattice in $\mathbb{C}$ is parabolic;

d) a finite, ramified covering of a parabolic Riemann surface is parabolic;

e) the unit disk *is not* parabolic.

f) Let X be a *compact* Riemann surface and $f\colon Y \to X$ be a Galois covering with automorphism group Γ. It is proven in [Gr] that Y is parabolic if and only if Γ has a subgroup of finite index isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ (or, of course, if Γ is finite).

The notion of parabolic Riemann surface have been introduced, (and studied), by Ahlfors (cf. [AS]); it is a class which strictly contains the class of algebraic Riemann Surfaces; never the less, Riemann surfaces in this class have many properties similar to the algebraic. Unfortunately the name can be misleading: in the uniformization theory of Riemann surfaces parabolic Riemann surfaces are those having as universal covering the complex plane; here this is not the case.

We recall here part of the theory and some of the properties of parabolic Riemann surfaces; for the proofs see [AS] and [SN].

Let M be a non compact parabolic Riemann surface and $M^* = M \cup \{\infty\}$ be its Alexandroff compactification.

4.3 Definition. *a) A* positive singularity *on M is a couple $(\tau; U)$, where U is a neighborhood of ∞ (so, by definition, $M \setminus U$ is compact) and τ is a positive harmonic function on U such that*

i) $\lim_{z\to\infty} \tau(z) = +\infty$;

ii) $\int_{\partial U} d^c\tau = -1$.

b) Two positive singularities $(\tau; U)$ and $(\tau'; U')$ are said to be equivalent *if $\tau - \tau'$ is a (upper and lower) bounded harmonic function on $U \cap U'$.*

We denote by $PS(M)$ the set of equivalence classes of positive singularities on M.

The relation between positive singularities and parabolic Riemann surfaces is the following:

4.4 Proposition. *The set $PS(M)$ is non empty.*

It is even possible to prove that the fact that $PS(M)$ is non empty is *equivalent* to the fact that M is parabolic.

4.5 *Examples.* a) If $M = \mathbb{C}$ then a positive singularity on M is $\tau(z) = \log |z|^2$ (with $U = \{|z| > 1\}$).

b) If $\pi : M \to \mathbb{C}$ is a finite ramified covering then $\tau = \frac{\pi^*(\log|z|^2)}{\deg \pi}$ (and $U = \pi^{-1}(|z| > 1)$) is a positive singularity on M.

Given a positive singularity $(\tau; U)$ on M we can define the Evans Kernel of it:

4.6 Definition. *A τ– Evans kernel is a function $e : M \times M \to (-\infty; \infty]$ such that*

i) $-e(z; q)$ is subharmonic in M as a function of z and harmonic in $M \setminus \{q\}$;

ii) for every $q \in M$ we can find an open disk Δ_q neighborhood of q such that

$$e(z; q)|_{\Delta_q} = -\log |z - q|^2 + \varphi$$

with φ harmonic, $e(z; q) \geq 0$ and $\int_{\partial\Delta_q} d^c_z(e(z; q)) = -1$.

iii) for every fixed $q \in M$, the function $-e(z; q)$ is a positive singularity on M and it is equivalent to τ;

iv) It is a symmetric function: $e(z; q) = e(q; z)$.

Remark that condition (iii) implies that, for a fixed q, we have that $\lim_{z\to\infty} -e(z, q) = +\infty$. Moreover, for every couple q_1 and $q_2 \in M$ the two positive singularities $-e(z; q_i)$ are equivalent; in particular there is a neighborhood of the infinity where $|e(z; q_1) - e(z, q_2)|$ is uniformly bounded.

This definition will be useless without the:

4.7 Proposition. *Given a positive singularity τ on M there exists a τ–Evans Kernel unique up to a constant.*

For the proof see [SN] page 355 and ff.

We remark that the existence of the Evans kernel implies, in particular, that, given a positive singularity τ and a point $q \in M$; there exists a positive singularity τ' equivalent to τ and such that the implied open set is $M \setminus \{q\}$.

Before we quote the regularity properties of the Evans kernel, we recall the definition of the *Green functions*:

4.8 Definition. *Let U be a regular region on a Riemann surface M and $P \in U$. A* Green function *for U and P is a function $g_{U;P}(z)$ on U such that:*

a) $g_{U;P}(z)|_{\partial U} \equiv 0$ continuously;

b) $dd^c g_{U;P} = 0$ on $U \setminus \{P\}$;

c) near P, we have $g_{U;P} = -\log|z-P|^2 + \varphi$, with φ continuous around P.

We easily deduce from the definitions that $dd^c g_{U;P} + \delta_P = \mu_{\partial U;P}$ where δ_P is the Dirac at P and $\mu_{\partial U;P}$ is a positive measure of total mass one and supported on ∂U.

Moreover the following is true:

4.9 Proposition. *The Green function, if it exists, it is unique.*

We have the following regularity properties of the Evans kernels:

4.10 Proposition. *a) The Evans Kernel is a continuous map $e : M \times M \to (-\infty;\infty]$. Denote by $\Delta \subset M \times M$ the diagonal; the Evans Kernel e is continuous and bounded on every open and relatively compact set $U \subset M \times M$, such that $\overline{U} \cap \Delta = \emptyset$.*

b) For every relatively compact open set $V \subset M$, we have a decomposition $e(p;q) = g_{V;q}(p) + v_{V;q}(p)$ with $v_{V;q}(\cdot)$ continuous and bounded.

c) Let $q_0 \in M$ and $M_\lambda = \{z \in M \,/\, e(z;q_0) \geq -\lambda\}$; then

$$e(p;q_0) = \lim_{\lambda\to\infty}(g_{M_\lambda;q_0}) - \lambda)$$

uniformly on every compact subset of $M \times M$, i.e. for every $K \subset M$ compact, we have

$$\lim_{\lambda\to\infty} \sup_{(p;q)\in K^2} |e(p;q) - (g_{M_\lambda;q} - \lambda)| = 0.$$

4.11 *Analytic maps of finite order from parabolic Riemann surfaces.*

In this part we will describe the main definitions and properties of the theory of analytic maps of finite order form a parabolic Riemann surface to a projective variety. This theory is similar to the classical Nevanlinna theory on maps from $\mathbb{C}$ to a projective variety. Since we did not find an adequate reference we will give some details.

We fix a parabolic Riemann surface M and a positive singularity $(\tau; U)$ on it (or more precisely its class in $PS(M)$). We also fix a closed set $F \subset U$ such that:

i) the interior of F is non empty;

ii) $\overline{M \setminus F}$ is compact.

Let X be a projective variety and $\overline{L}$ be an ample line bundle equipped with a C^∞ positive metric and let $\gamma : M \to X$ be an analytic map.

For every $t \in \mathbb{R}$ we define the *characteristic function* $T_{\gamma,[\tau]}(t)$ of γ with respect to the class $[\tau]$ in the following way:

$$T_{\gamma;[\tau]}(t) := \int_F (t-\tau)^+ \gamma^*(c_1(\overline{L})),$$

(where $(f(x))^+ := \sup\{f(x); 0\}$).

We will say that γ has *finite order* $\rho \in \mathbb{R}_{\geq 0}$ with respect to $[\tau]$ if

$$\limsup_{t\to\infty} \frac{\log T_{\gamma;[\tau]}(t)}{t} = \rho.$$

This is equivalent to say that, for every $\epsilon > 0$ and $t \gg 0$ we have $T_{\gamma;[\tau]}(t) \leq \exp(t(\rho+\epsilon))$. Making the change of variables $t = \log r$, we see that γ is of finite order ρ if, as soon as $r \to \infty$, $T_{\gamma;[\tau]}(\log r) \leq r^{\rho+\epsilon}$.

4.11 *Example.* When $M = \mathbb{C}$ and $\tau(z) = \log|z|^2$ then $T_{\gamma;[\tau]}(\log r)$ is the classical characteristic function, defined, for instance, in [GK].

We prove now that the order of a map depends only on the map itself and on the class of τ in $PS(M)$. We introduce now the, so called, non integrated version of the characteristic function:

Let M, τ, F, γ etc. as before. If t is sufficiently big, we will introduce the following sets: $F[t] := \{z \in F \,/\, \tau(z) \leq \log(t)\}$ and $B(r) := \{(z;y) \in F \times \mathbb{R} \,/\, \tau(z) \leq y \,\text{and}\, y \leq \log(r)\}$. From the definitions we obtain:

$$\begin{aligned} T_{\gamma;[\tau]}(\log(r)) &= \int_F (\log(r) - \tau(z))^+ \gamma^*(c_1(\overline{L})) \\ &= \int_{F[r]} (\int_{\tau(z)}^{log(r)} dy) \gamma^*(c_1(\overline{L})). \end{aligned}$$

If we apply Fubini theorem we find that this last integral is

$$\begin{aligned} \int_{B(r)} dy \wedge p_1^*(\gamma^*(c_1(\overline{L}))) &= \int_{-\infty}^{log(r)} dy \int_{F[e^y]} \gamma^*(c_1(\overline{L})) \\ &= \int_0^r \frac{dt}{t} \int_{F[t]} \gamma^*(c_1(\overline{L})). \end{aligned}$$

Call the function $t_{\gamma;[\tau]}(y) = \int_{F[y]} \gamma^*(c_1(\overline{L}))$, the *non integrated characteristic function* of γ with respect to $[\tau]$, and we obtain

$$T_{\gamma;[\tau]}(\log(r)) = \int_0^r t_{\gamma;[\tau]}(y) \frac{dy}{y}.$$

4.12 *Example.* It is well known that, if $M = \mathbb{C}$ and $\tau = \log|z|^2$, then a possible definition of the characteristic function is

$$T_\gamma(r) = \int_0^r \frac{dt}{t} \int_{|z|\leq t} \gamma^*(c_1(\overline{L})).$$

We are now ready to prove the independence of the order:

4.13 Theorem. *The order ρ of an analytic map γ from a parabolic Riemann surface M to a projective variety X depends only on γ and on the class $[\tau] \in PS(M)$. more precisely it is independent on:*

a) the choice of the closed set F;

b) the choice of the representative $\tau \in [\tau]$;

c) the choice of the metrized ample line bundle $\overline{L}$ on X.

Proof: a) Independence on F:given two closed sets F and F', then, if the closure of the complementary of F and F' are compact then also the closure of the complementary of $F \cap F'$ is also compact; consequently we may suppose that $F' \subset F$. Let $F' \subset F$ with F and F' as before. We denote by $T_\gamma^F(t)$ ($T_\gamma^{F'}(t)$) and by ρ^F ($\rho^{F'}$ the characteristic function and the order respectively, computed by using F (F'). Since $T_\gamma^F(t) \geq T_\gamma^{F'}(t)$, then $\rho^F \geq \rho^{F'}$.

On the other direction; denote $\|\tau\|_{L^\infty(F\setminus F')}$ the sup of τ over $F \setminus F'$; remark that $\|\tau\|_{L^\infty(F\setminus F')} < \infty$ because the closure of $F \setminus F'$ is compact. Thus we have:

$$\begin{aligned} 0 \leq T_\gamma^F(t) - T_\gamma^{F'}(t) &= \int_{F\setminus F'} (t-\tau)^+ \gamma^*(c_1(\overline{L})) \\ &\leq (|t| + \|\tau\|_{L^\infty(F\setminus F')}) \int_{F\setminus F'} \gamma^*(c_1(\overline{L})) \end{aligned}$$

Consequently we can find a constant $A \in \mathbb{R}_{\geq 0}$ such that $T_\gamma^F(t) \leq T_\gamma^{F'}(t) + A|t|$. Suppose t (and consequently $T_\gamma^F(t)$ and $T_\gamma^{F'}(t)$) sufficiently big (otherwise ρ^F and $\rho^{F'}$ are both zero); we have

$$\begin{aligned} \rho^F = \limsup_{t\to\infty} \frac{\log(T_\gamma^F(t))}{t} &\leq \limsup_{t\to\infty} \frac{\log(T_\gamma^{F'}(t) + A|t|)}{t} \\ &\leq \limsup_{t\to\infty} \frac{\log(T_\gamma^{F'}(t)) + \log(A|t|)}{t} = \rho^{F'}. \end{aligned}$$

b) Independence on $\tau \in [\tau]$: First of all it is easy to see that if $\tau' = \tau + A$ (for a suitable constant $A \in \mathbb{R}$), then, if we compute the order of γ by using τ' or by using τ we obtain the same number. In general, if τ_1 and τ_2 are in the same class in $PS(M)$, then, over suitable open sets, we can suppose that there exist two constants A and B such that $A \leq \tau_2 - \tau_1 \leq B$. Consequently we can suppose that $\tau_1 \leq \tau_2$. Put $F_i[y] := \{\tau_i(z) \leq \log y\}$ ($i = 1, 2$). Then, for $y \gg 0$, $F_2[y] \subseteq F_1[y]$, consequently,

one easily see, by using the non integrated form of the characteristic function, that $T_{\gamma,\tau_1}(t) \geq T_{\gamma,\tau_2}(t)$. From this we conclude.

c) Independence on $\overline{L}$: First of all we check that the definition of the order is independent on the choice of the metric on the ample line bundle L: Suppose that we choose two (positive) metrics $\| \cdot \|_1$ and $\| \cdot \|_2 = \| \cdot \|_1 \cdot \exp(\varphi)$ on L, with φ a bounded C^∞ function on X. If we denote by $T_{\gamma,i}(t)$ the characteristic function computed by using the metric $\| \cdot \|_i$, we have

$$T_{\gamma,2}(\log(r)) = T_{\gamma,1}(\log(r)) + \int_0^r \frac{dy}{y} \int_{F[y]} \gamma^* dd^c \varphi.$$

By Stokes Theorem we have

$$\begin{aligned}\int_0^r \frac{dy}{y} \int_{F[y]} \gamma^* dd^c \varphi &= \int_0^r \frac{dy}{y} \int_{\partial F[y]} \gamma^* d^c \varphi \\ &= \int_{F[e^r]} d\tau \wedge \gamma^* d^c \varphi.\end{aligned}$$

Thus since τ is harmonic, the following equality holds: $d\tau \wedge \gamma^* d^c \varphi = d\gamma^* \wedge d^c \tau = d(\gamma^* \varphi \cdot d^c \tau)$. By applying Stokes again, we obtain that the last integral is equal to

$$\int_{\partial F[e^r]} \gamma^* \varphi \cdot d^c \tau.$$

We deduce the independence on the metrics because of property (ii) of 4.3 and the fact that φ is bounded.

In order to prove that the order of γ is independent on the choice of the ample line bundle L, it is sufficient to remark that, if L_1 and L_2 are two ample line bundles on X, such that $L_1 \otimes L_2^{-1}$ is ample then, for $t \gg 0$, $T_{\gamma;L_1}(t) \geq T_{\gamma;L_2}(t)$, and, for every positive integer D we have $T_{\gamma;L_1^D}(t) = DT_{\gamma;L_1}(t)$.

4.14 *Points on the image of maps of finite order.*

Let M be a (non compact) parabolic Riemann surface and τ a positive singularity on M. Let $e(p;q)$ be the τ–Evans Kernel over M. We can define a "canonical" metric on the tangent bundle TM and, for every point $p \in M$ on the line bundle $\mathcal{O}_M(p)$: Let Δ be the diagonal divisor on $M \times M$; we put a metric on the line bundle $\mathcal{O}_{M\times M}(\Delta)$ in the following way: if $\mathbb{I}_\Delta$ is the section of $\mathcal{O}_{M\times M}(\Delta)$ defining Δ, we define $\|\mathbb{I}_\Delta\|(p;q) := \exp(-e(p;q))$. Observe that, by property (ii) of the Evans Kernel (definition 4.6), the metric on $\mathcal{O}_{M\times M}(\Delta)$ is well defined.

By pull–back, we define a metric on $TM := \Delta^*(\mathcal{O}_{M\times M}(\Delta))$; for every point $p \in M$ let $\beta_p : M \to M \times M$ is the map $\beta_p(q) = (p;q)$; on $\mathcal{O}_M(p)$ we put the metric induced by the isomorphism $\mathcal{O}_M(p) = \beta_p^*(\mathcal{O}_{M\times M}(\Delta))$.

4.14 Remark. If we choose the "other" embedding $\alpha_p(q) := (q;p)$ we obtain the same

metric on $\mathcal{O}_M(p)$ because of the symmetry of the Evans Kernel.

By construction, for every point $p \in M$ we find a canonical adjunction isometry

$$TM|_p \simeq \mathcal{O}_M(p)|_p.$$

Let X be a projective variety, $\overline{L}$ a metrized ample line bundle over X (with positive metric) and $\gamma : M \to X$ be an analytic map of finite order ρ with respect to $[\tau]$. We suppose that $\gamma(M)$ is Zariski dense in X. For every positive integer, we have an injective map

$$\gamma^* : H^0(X; L^D) \hookrightarrow H^0(M; \gamma^*(L^D)).$$

We fix a finite set of points $F := \{p_1; \ldots; p_s\} \subset M$. and, for every positive integer i, we consider the canonical injective map

$$\eta_i : H^0(M; \gamma^* L^D(-i \sum_{p_j \in F} p_j)) \hookrightarrow H^0(M; \gamma^* L^D).$$

We denote by E_D^i the subspace $(\gamma^*)^{-1}(\eta_i(H^0(M; \gamma^* L^D(-i \sum_{p_j \in F} p_j))))$ and by γ_D^i the composite map

$$\begin{aligned}
&E_D^i \xrightarrow{\gamma^*} \eta_i(H^0(M; \gamma^* L^D(-i \sum_{p_j \in F} p_j))) \\
&\longrightarrow \eta_i(H^0(M; \gamma^* L^D(-i \sum_{p_j \in F} p_j)))/\eta_{i+1}(H^0(M; \gamma^* L^D(-(i+1) \sum_{p_j \in F} p_j))) \\
&\simeq \bigoplus_{p_j \in F} (\gamma^*(L^D) \otimes (TM)^{-i})|_{p_j};
\end{aligned}$$

the E_D^i's are finite dimensional vector spaces equipped with the sup norm. The vector space $\bigoplus_{p_j \in F} (\gamma^*(L^D) \otimes (TM)^{-i})|_{p_j}$ is also a finite dimensional hermitian vector space: we equip it with the "direct sum" metric. Eventually, if $p_h \in F$, we let $\gamma_{D,h}^i : E_D^i \to \gamma^*(L^D) \otimes (TM)^{-i})|_{p_h}$ be the linear map composite of γ_D^i with the canonical projection $\bigoplus_{p_j \in F} (\gamma^*(L^D) \otimes (TM)^{-i})|_{p_j} \to \gamma^*(L^D) \otimes (TM)^{-i})|_{p_h}$.

The main theorem of this chapter is the following: if, in Theorem 2.3, the involved analytic germs come from a parabolic Riemann surface and the corresponding map is of finite order, then we can prove strong version of Schwartz lemma. More precisely:

4.15 Theorem. *Let $\epsilon > 0$. Under the hypotheses above, there is a constant C depending only on γ, on F etc., but independent on i and D (provided that $\frac{i}{D}$ is sufficiently big), such that, for every h, we have*

$$\log \|\gamma_{D,h}^i\| \leq -\frac{i \cdot Card(F)}{\rho + \epsilon} \log(\frac{i}{D}) + i \cdot C.$$

Proof: Let $p := p_h$. Let r be a sufficiently big positive real number such that

$$\Omega_r := \{z \in M \,/\, \|\mathbb{I}_p\| \leq r\} \supset F;$$

thus, by definition $\Omega_r = \{z \in M \,/\, -e(z;p) < \log(r)\}$. On Ω_r we consider the function $g_{r;p}(z) := \log(r) + e(z;p)$; this function has the following properties:

– near p it is of the form $-\log|z-p|^2 + \varphi_p(z)$ with $\varphi_p(z)$ harmonic near p;

– $\lim_{z\to\partial\Omega_r} g_{r;p} = 0$ uniformly;

– $g_{r;p}$ is continuous and harmonic in $\Omega_r \setminus \{p\}$;

– it is positive.

If we define $g_{r;p} \equiv 0$ on $M \setminus \Omega_r$ we can extend $g_{r;p}$ to a continuous function on M, which we will denote again by $g_{r;p}$. By construction $g_{r;p}$ is the Green function of p in Ω_r (cf. 4.9). In particular $dd^c g_{r;p} = -\delta_p + \mu_{p;\partial\Omega_r}$ (as distributions), where $\mu_{p;\partial\Omega_r}$ is a positive measure of total mass one and supported on $\partial\Omega_r$.

Let $s \in E^i_D$; thus, by definition, $\gamma^*(s) = \eta_i(\tilde{s})$, where $\tilde{s}$ is a global section of $\gamma^*(L^D)(-i\sum P_j)$. We will denote by $\|s\|$ the norm of s as section of the hermitian line bundle $\gamma^*(L^D)$ and by $\|\tilde{s}\|$ the norm of $\tilde{s}$ as section of the hermitian line bundle $\gamma^*(L^D)(-i\sum P_j)$. By Stokes Theorem we have

$$\int_M \log\|\tilde{s}\|^2 \cdot dd^c g_{r;p} = \int_M dd^c \log\|\tilde{s}\|^2 \cdot g_{r;p}.$$

Consequently we get

$$\int_M \log\|\tilde{s}\|^2(-\delta_p + \mu_{p;\partial\Omega_r}) = \int_M (\delta_{div(\tilde{s})} - D\gamma^*(c_1(\overline{L})) \cdot g_{r;p};$$

where $\delta_{div(\tilde{s})}$ is the Dirac measure supported on the effective "divisor" of the zeros of $\tilde{s}$. Since $\mu_{p;\partial\Omega_r}$ is positive of total mass one and $g_{r;p}$ is positive we obtain

$$-\log\|\tilde{s}\|^2(p) + \log\|s\|^2_{L^\infty} + i\sum_{p_j\in F}\int_M e(\cdot;p_j)\mu_{p;\partial\Omega_r} \geq -D\int_M \gamma^*(c_1(L))\cdot(\log r + e(\cdot;p))^+.$$

By property (iii) of the Evans Kernel, (cf. 4.6) we then obtain

$$\log\|s\|^2_{L^\infty} + i\sum_{p_j\in F}\int_M e(\cdot;p_j)\mu_{p;\partial\Omega_r} + DT_{\gamma;[\tau]}(\log(r)) \geq \log\|\tilde{s}\|^2(p).$$

By applying Prop 4.10 there exists a constant B and an open set $U \subseteq M$ such that $M \setminus U$ is compact, contains F and, for every $p_j \in F$ and for $z \in U$ we have $|e(z;p_j) - e(z;p)| \leq B$ (they both define equivalent positive singularities). So on $\partial\Omega_r$ (which, for $r \gg 0$ we can suppose contained in U) we have $|\log(r) + e(z;p_j)| \leq B$; from

this we get

$$\log \|s\|^2_{L^\infty} - i\cdot Card(F)\log(r) + DT_{\gamma;[\tau]}(\log(r)) + i\cdot B_1 \geq \log\|\tilde{s}\|^2(p).$$

By hypothesis γ is of finite order ρ, thus, for $r \gg 0$, we have $T_{\gamma;[\tau]}(\log(r)) \leq r^{\rho+\epsilon}$, consequently

$$\log \|s\|_{L^\infty} - i\cdot Card(F)\log(r) + Dr^{\rho+\epsilon} + i\cdot B_1 \geq \log\|\tilde{s}\|(p).$$

Let $h(r) = -i\cdot Card(F)\log(r) + Dr^{\rho+\epsilon}$; we have $\frac{dh}{dr}(r) = -\frac{i\cdot Card(F)}{r} + (\rho+\epsilon)Dr^{\rho+\epsilon-1}$; thus $h(r)$ has a minimum in $r_0 = \left(\frac{i\cdot Card(F)}{(\rho+\epsilon)D}\right)^{\frac{1}{\rho+\epsilon}}$, which, as soon as $\frac{i}{D}$ is sufficiently big is allowed. In this case

$$h(r_0) = -\frac{i\cdot Card(F)}{\rho+\epsilon}\log(\frac{i}{D}) + i\cdot B_2$$

with B_2 independent on i and D. From this we conclude.

Now we can state, and prove the main theorem of this section. As in the introduction, we suppose that K is a number field and $\sigma_0 : K \hookrightarrow \mathbb{C}$ is an embedding of K in $\mathbb{C}$. We also fix an embedding of the algebraic closure $\overline{K}$ of K in $\mathbb{C}$. Let X be a quasi projective variety of dimension N defined over K; Let $S \subseteq X(\overline{K})$ and, for every positive integer r denote by S_r the set $\{x \in S \ s.t.\ [\mathbb{Q}(x) : K] \leq r\}$.

4.16 Theorem. *Let M be a parabolic Riemann surface (with a fixed positive singularity). Let $\gamma\colon M \to X(\mathbb{C})$ be an holomorphic map of finite order ρ with Zariski dense image. Suppose that, for every $\overline{K}$–rational point $P \in S\cap\gamma(M)$, the formal germ $\hat{M}_P$, of M near P, is (the pull back of) a LG– germ of type α (in its field of definition). then*

$$\frac{Card(\gamma^{-1}(S_r))}{r} \leq \frac{N+1}{N-1}\rho\alpha[K:\mathbb{Q}].$$

Proof: It suffices to prove that, given a finite set of points $F = \{P_1,\ldots,P_s\} \subset \gamma^{-1}(S_r)$ and $\epsilon > 0$, we have

$$\frac{Card(F)}{r} \leq \frac{N+1}{N-1}(\rho+\epsilon)\alpha[K:\mathbb{Q}].$$

Let F be such a set and L be a finite extension of K which contains $\mathbb{Q}(\gamma(P_j))$ for every $P_j \in F$.

For every number field L we will denote by M_L the set of places of L. We fix P_j and, and let $\sigma_{P_j} \in S_{\mathbb{Q}(P_j)}$ be the embedding extending σ_0. If $i/D \gg 0$, from Corollary 3.13 and Theorem 4.15 we get,

$$
\begin{aligned}
\frac{1}{[\mathbb{Q}(P_j):\mathbb{Q}]} & \sum_{\sigma\in M_{\mathbb{Q}(P_j)}} \log\|\gamma^i_{D,j}\|_\sigma \\
&= \frac{1}{[\mathbb{Q}(P_j):\mathbb{Q}]}\left(\log\|\gamma^i_{D,j}\|_{\sigma_{\sigma_{P_j}}} + \sum_{\sigma\neq\sigma_{P_j}} \log\|\gamma^i_{D,j}\|_\sigma\right) \\
&\leq -\frac{Card(F)}{(\rho+\epsilon)\cdot[\mathbb{Q}(P_j):\mathbb{Q}]} i\log\left(\frac{i}{D}\right) + \alpha\cdot i\log(i) + C_2\,(i+D)),
\end{aligned}
$$

with C_2 independent on i, D and $\mathbb{Q}(P_j)$. Thus, we deduce

$$
\begin{aligned}
\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma\in M_L}\log\|\gamma^i_D\|_\sigma &\leq \frac{1}{[L:\mathbb{Q}]}\sum_{\sigma\in M_L}\sup_{P_j\in F}\log\|\gamma^i_{D,j}\|_\sigma + \log(Card(F)) \\
&\leq -\frac{Card(F)}{r\cdot[K:\mathbb{Q}](\rho+\epsilon)} i\log\left(\frac{i}{D}\right) + \alpha\cdot i\log(i) + C_2\,(i+D) + \log(Card(F)).
\end{aligned}
$$

But since the image is Zariski dense and the dimension of X is N, we can apply Theorem 2.3 and we obtain that

$$
\frac{Card(F)}{r\cdot[K:\mathbb{Q}]} \leq \frac{N+1}{N-1}(\rho+\epsilon)\alpha.
$$

and we eventually conclude.

We state also the following corollaries, which show the the link with the classical Schneider–Lang Theorem (where $X=\mathbb{C}^n$ and $M=\mathbb{C}$):

4.17 Corollary. *Suppose we are in the hypotheses above, then*

$$
\sum_{P\in\gamma^{-1}(S)}\frac{1}{[\mathbb{Q}(P);K]} \leq \frac{N+1}{N-1}\rho\alpha.
$$

If X is a quasi projective variety defined over the number field K and r is a positive integer, denote by X_r the set $\{P\in X(\overline{K})\ \text{ s.t. }\ [\mathbb{Q}(P):K]\leq r\}$.

4.18 Corollary. *Let X be an algebraic variety defined over a number field K and let $F\hookrightarrow T_X$ be a foliation of rank one (defined over K). Suppose that the holomorphic foliation $F_\sigma\subset(T_X)_\sigma$ has a leaf M which is parabolic of finite order ρ (for some positive singularity on M) whose Zariski closure has dimension $d>1$, then*

$$
\frac{Card((X_r\setminus Sing(F))\cap M)}{r} \leq \frac{d+1}{d-1}\rho[K:\mathbb{Q}].
$$

4.19 Corollary. *If, in the hypotheses of 4.18, the foliation is closed under p derivation for almost all primes $\mathfrak{p}$ of O_K, then the leaf passing through an algebraic point is an algebraic curve.*

The proofs of the corollaries are straightforward applications of the main theorem.

4.20 Remark. One should notice that corollary 4.19 is a particular case of the main theorem of [Bo].

5 Rational points on higher dimensional subvarieties.

In this section we will deal with Khäler varieties which enjoy properties similar to the parabolic Riemann surfaces. We will study analytic maps between these varieties and Projective varieties. When one look carefully to the previous section one sees that the main tool to prove the main theorems was the existence of the Evans kernel on the Riemann surface we are working with. We will see that The analogous of a Evans kernel will suffice to develop a value distribution theory and the notion of the order of an analytic map.

5.1 Definition. *Let A be a d dimensional Khäler manifold with a fixed Khäler metric ω. We will say that (A, ω) (or simply A) is* conformally parabolic *if there exists a function*

$$g : A \times A \longrightarrow (-\infty; \infty]$$

with the following properties:

a) For every $p \in A$ the function $g_p(z) := g(p, z)$ is C^∞ in $A \setminus \{p\}$ and satisfy the following differential equation:

$$dd^c(g_p) \wedge \omega^{d-1} = \delta_p;$$

where δ_p is the Dirac measure concentrated in p;

b) for every p we have that

$$\lim_{z \to \infty} g_p(z) = +\infty;$$

c) for every couple p and q there exists a neighborhood of the infinity U and a constant C such that, for every $z \in U$

$$|g_p(z) - g_q(z)| \leq C.$$

One easily sees that if $d = 1$, up to the sign, g is an Evans kernel.

We will call the function g, a *Evans Kernel* of the conformally parabolic manifold A.

Before we develop a value distribution theory on conformally parabolic varieties, we give the main example we have in mind.

5.2 *Examples of conformally Parabolic varieties.* Let $\overline{A}$ be a compact Khäler manifold of dimension d, ω a Khäler form on it and H_∞ be a divisor on $\overline{A}$. Let A be the open

set $\overline{A}\setminus H_\infty$. We will denote by R the degree of H_∞ with respect to ω, we also fix a C^∞ metric $\|\cdot\|_{H_\infty}$ on $\mathcal{O}_{\overline{A}}(H_\infty)$.

Let $p \in A$ be a point.

5.2 Theorem. *There exists a function $g_p\colon \overline{A} \to [-\infty, +\infty]$ unique up to an additive constant with the following properties:*

(i) g_p is C^∞ outside $\{p\} \cup H_\infty$;

(ii) g_p is a solution of the following differential equation:

$$dd^c(g_p) \wedge \omega^{d-1} = \delta_p - \frac{1}{R}\delta_{H_\infty}\omega^{d-1};$$

δ_p (resp. δ_{H_∞}) being the Dirac measure concentrated on p (resp. H_∞).

We will only sketch the proof of 5.2. Details can be filled by standard Hodge theory, cf. for instance [Vs].

Proof: Let T be the current $\delta_p - \frac{1}{R}\delta_{H_\infty}\omega^{d-1}$. It is a current of bi–degree (d,d) and $\int_{\overline{A}} T = 0$. Consequently, by the Hodge decomposition, there exists a current S such that $\Delta_{\overline{\partial}}(S) = T$ ($\Delta_{\overline{\partial}}$ being the laplacian with respect to the metric ω). Since T is C^∞ outside $\{P\} \cup H_\infty$, one can prove, by following the proof of 6.32 chapter 6 of [Wa] that also the current S is C^∞ outside $\{P\} \cup H_\infty$. Let L be the operator on forms obtained by wedging with the form ω. Denoting by $\mathcal{D}^{i,i}$ the sheaf of currents of bidegree (i,i), it is well known that L^d induces an isomorphism between $\mathcal{D}^{0,0}$ and $\mathcal{D}^{d,d}$. Thus, there is a distribution $\tilde{g}_p$ such that $L^d(\tilde{g}_p) = S$. By the standard commutation rules between L, $\Delta_{\overline{\partial}}$, ∂, $\overline{\partial}$ and $\overline{\partial}^*$, we can find a constant c (which depends only on d) such that, for every local section $g \in \mathcal{A}^{0,0}$,

$$dd^c(g) \wedge \omega^{d-1} = c \cdot \Delta_{\overline{\partial}}(g) \wedge \omega^d = c \cdot L^d(\Delta_{\overline{\partial}}(g)).$$

Moreover $L^d(\Delta_{\overline{\partial}}(g)) = \Delta_{\overline{\partial}}(L^d(g))$. Consequently we can find a function g_p with the properties claimed by the statement. The difference of two functions verifying (i) and (ii) will be an harmonic function on $\overline{A}$, thus it will be a constant.

5.3 Theorem. *The function $g(p,z) := g_p(z)$ is an Evans kernel on A.*

Proof: Let $s_\infty \in H^0(\overline{A}, \mathcal{O}_{\overline{A}}(H_\infty))$ be a section such that $div(s_\infty) = H_\infty$; then we claim that there exists a C^∞ function f_p on $\overline{A}\setminus\{p\}$ such that, on $\overline{A}\setminus\{p\}$,

$$g_p = -\frac{1}{R}\log\|s_\infty\|^2_{H_\infty} + f_p.$$

Indeed by Poincaré–Lelong formula, $dd^c \log\|s_\infty\|^2_{H_\infty} = \delta_{H_\infty} - c_1(\mathcal{O}_{\overline{A}}(H_\infty); \|\cdot\|_{H_\infty})$, we have that, on $\overline{A}\setminus\{p\}$

$$dd^c(g_p + \frac{1}{R}\log\|s_\infty\|^2_{H_\infty}) \wedge \omega^{d-1} = -\frac{1}{R}c_1(\mathcal{O}_{\overline{A}}(H_\infty); \|\cdot\|_{H_\infty}) \wedge \omega^{d-1};$$

part the claim follows by the same method of the proof of 5.2. Consequently properties (a), (b) and (c) are easily verified.

5.4 Corollary. *The complementary of an effective divisor of a compact Khäler manifold is conformally parabolic. In particular every quasiprojective variety is conformally parabolic.*

5.6 *counting functions.*

In this subsection we will develop a value distribution theory for conformally parabolic manifold which is analogue to the value distribution theory on parabolic Riemann surfaces.

We fix a d dimensional ($d \geq 2$)) conformally parabolic manifold A with Khäler form ω and Evans kernel g (we suppose that $d \geq 2$ because the case $n = 1$ is treated in the previous section and it is a little bit different on the estimates).

Before we need to state and prove some of the properties of the functions g_p's.

5.6 Proposition. *Let $p \in A$. There exists a neighborhood U_p of p, analytically equivalent to a d–th dimensional ball centered at p (with coordinates z) and a continuous non vanishing function χ_p, such that*

$$|g_p||_{U_p} \cdot \|z\|^{2(d-1)} = \chi_p(z).$$

Moreover there is a positive constant α such that

$$\chi_p(z) = \frac{1}{n-1} + O_\omega(\|z\|^\alpha).$$

Proof: the stated properties can be deduced from Theorem 4.13 page 108 of [Au]; more precisely from formula (17) page 109, Lemma 4.12 page 107 and formula (8) page 106 of loc. cit.

5.7 Remark. One can deduce from the proof of Theorem 4.13 page 109 of [Au] that $\chi_p(z)$ is a function which is regular enough for our purposes. Indeed it will be of class $C^{1,\alpha}$ for every $\alpha < 1$.

For every continuous function $\lambda : A \to [-\infty, +\infty)$ we define the following sets:

$$S_\lambda(t) := \{z \in A \text{ such that } \lambda(z) = t\}\,;$$

and

$$B_\lambda(t) := \{z \in A \text{ such that } \lambda(z) \leq t\}\,.$$

Moreover, if D is a subset of A, we denote by $D_\lambda(t)$ the set $D \cap B_\lambda(t)$.

Remark that, by condition (b)of the definition of the Evans kernel, if $p \in A$, S_{g_p} and B_{g_p} are compact.

If α is a $(1,1)$ form on A we define the *counting function* with respect to α to be

$$\begin{aligned} T_\alpha(r) &:= \int_0^r \frac{dt}{t} \int_{B_{g_p}(\log(t))} \alpha \wedge (\omega)^{d-1} \\ &= \int_{-\infty}^{\log(r)} dt \int_{g_p \le t} \alpha \wedge (\omega)^{d-1} \\ &= \int_A (\log(r) - g_p)^+ (\alpha \wedge (\omega)^{d-1}) \end{aligned}$$

(where, as in §4, $(f)^+$ means $\sup\{f, 0\}$).

Let X be projective variety and $\gamma\colon A \to X$ be an analytic map. Suppose that $\overline{L}$ is an ample line bundle equipped with a positive metric. We define $T_\gamma(r)$ to be $T_{\gamma^*(c_1(\overline{L}))}(r)$.

5.8 Definition. *We say that $\gamma\colon A \to X$ is of finite order ρ if*

$$\limsup_{r\to\infty} \frac{\log(T_\gamma(r))}{\log(r)} = \rho.$$

As in the previous section, this means that, for every $\epsilon > 0$ there is a r_0 such that for $r \ge r_0$ we have

$$T_\gamma(r) \le r^{\rho+\epsilon}.$$

The order depends only on the map γ; in particular it is independent on the choice of the ample line bundle L (and on the metric on it).

5.9 Proposition. *The order of $\gamma : A \to X$ depends only on γ; more precisely:*

a) the order is independent on the choice of the point p;

b) the order is independent on the choice of the ample line bundle with positive metric L on X.

Proof: Let p_1 and p_2 be two points on A. By property (c) of the Evans kernel, we can find a compact set K containing the p_i's and constants C_i such that, outside K, $g_{p_2} + A_2 \le g_{p_1} \le g_{p_2} + C_1$. Consequently the proof is analogous to the case of parabolic Riemann surfaces and we leave the details to the reader.

We will show now that we can use the counting function $T(r)$ in order to control the norm of jets of sections of line bundles.

Let L be an hermitian line bundle equipped with a positive metric on A. Let $s \in H^0(A, L)$. We define the *proximity function* of s to be

$$\mu_p(\log\|s\|^2)(r) := \int_{S_{g_p}(\log(r))} \log\|s\|^2 \cdot d^c(g_p) \wedge \omega^{d-1}.$$

By Stokes theorem we have

$$\int_{S_{g_p}(\log(r))} d^c(g_p) \wedge \omega^{d-1} = 1$$

and, for $r \gg 0$, $d^c g_p \wedge \omega^{d-1}$ is a positive measure on $S_{g_p}(r)$. Consequently, if $\|s\|_\infty < \infty$ and $r \gg 0$,

$$\log \|s\|_\infty^2 \geq \mu_p(\log \|s\|^2)(r). \tag{5.10.1}$$

The following theorem is an analogue of the Nevanlinna F.M.T., in this contest.

5.11 Proposition. *For every R and r with $R > r$ the following equality holds:*

$$\mu_p(\log \|s\|^2)(R) - \mu_p(\log \|s\|^2)(r) = \int_{\log(r)}^{\log(R)} dt \int_{g_p \leq t} (\delta_{div(s)} - c_1(L)) \wedge (\omega)^{d-1}.$$

The proof is a direct application of Stokes formula as in the proof of the Nevanlinna F.M.T. (cf [GK]).

Let L be a line bundle on A equipped with a positive metric. Let $s \in H^0(A, L^D)$ be a global section of it. Suppose that s vanishes at the order i in p. Let $j^i(s)$ be its i–th jet in p. It is a well defined section of $S^i(T_A^*) \otimes L^D|_p$. Since we fixed a metric on A and on L, the vector space $S^i(T_A^*) \otimes L^D|_p$ is equipped with an hermitian metric; we will denote by $\| \cdot \|_{i,D}$ the induced norm on it.

We will give a bound of the norm of $j^i(s)$ in terms of $\mu_p(\log \|s\|^2)(R)$. This can be seen as an analogue of the classical Cauchy inequality on the complex plane.

5.12 Proposition. *There exists a constant C depending only on r and on the metric on L such that, for every $s \in H^0(A, L^D)$ as above, we have*

$$\mu_p(\log \|s\|^2)(r) \geq \log \|j^i(s)\|_{i,D}^2 + C(i + D).$$

Since this statement do not change if we change the metric on the tangent bundle T_A (but if we change it of course, the constant C will change), we are free to choose the metric we prefer on the vector space $T_A|_p$.

We consider the blow up $\tilde{A}$ of A in p let E be the exceptional divisor. We can define a metric on $\mathcal{O}(E)$ in the following way: $\|E\|(z) := \frac{1}{|g_p(z)|^{1/2(d-1)}}$. The Kahler metric ω induces a Fubini–Study metric ω_{FS} on E. Let $\tilde{s}$ be the strict transform of s; it is a section of the hermitian line bundle $L(-iE)$. We know that $\log \|j^i(s)\|_{i,D}^2 = \int_E \log \|\tilde{s}\|^2 \omega_{FS}^{d-1} + C$ where C is a controlled constant (cf. [Bo] §4.3.2). By Theorem 4.13 page 109 of [Au] one can easily show that

$$\lim_{R \to -\infty} \int_{g_p = R} \log \|\tilde{s}\|^2 d^c g_p \wedge \omega^{d-1} = \int_E \log \|\tilde{s}\|^2 \omega_{FS}^{d-1}.$$

Proposition 5.12 will be consequence of the following more general statement.

5.13 Theorem. *There exists a $\tilde{r} \gg 0$ and a $\gamma > 0$ depending only on the Kahler*

metric ω such that, if $\log(r) > \tilde{r}$ the following holds:

$$T_{c_1(L)}(r) \geq -\mu_p(\log\|s\|^2)(r) + \log\|j^i(s)\|_{i,D} + \int_{-\tilde{r}}^{\log(r)} dt \int_{div(s)_{g_p}(t)} \omega^{d-1} - \frac{i}{d-1}\cdot\left(\log(\tilde{r}) + O_\omega(\frac{1}{\tilde{r}^\gamma})\right).$$

Proof: We start with the following: Since the metric ω is Kähler, we can choose a neighborhood U_p of p, analytically equivalent to the unit ball and coordinates $(z_1, \dots z_d)$ centered at p on it, in such a way that the following property holds: denote by $\lambda = dd^c|z|^2$ the standard Euclidean (1,1) form on U_p; then $\omega|_{B_r} = \lambda + k$ with $k = O(|z|^2)$ when $|z| \to 0$.

5.14 Lemma. *Suppose that $0 < \tilde{r} < R$ and $\tilde{r}$ is such that $B_{g_p}(-\tilde{r}) \subseteq U_p$; then we can find a positive constant γ such that*

$$\int_{-R}^{-\tilde{r}} dt \int_{div(s)_{g_p}(t)} \omega^{d-1} \geq \frac{i}{d-1}\cdot\left(\log\frac{R}{\tilde{r}} + O_\omega(\frac{1}{\tilde{r}^{1/d-1}} + O_\omega(\frac{1}{R^\gamma}))\right);$$

The involved constants depend only on the point p, the metric ω and the Evans kernel (and they are independent on the section s).

Proof: By 5.6, we can find a constant a such that, in U_p, $|g_p|\|z\|^{2(d-1)} \geq (\frac{1}{d-1} - a\|z\|^\alpha)$. Thus, The ball $B_z(t) := \{\|z\|^{2(d-1)} \leq \frac{1}{t}(\frac{1}{d-1} - \frac{a}{t^\alpha})\}$ is contained in $B_{g_p}(-t)$. As a consequence of [GK] Lemma 1.16 and Prop. 1.17 we find that

$$\int_{div(s)_{g_p}(t)} \lambda^{d-1} \geq \int_{div(s)\cap B_z(|t|)} \lambda^{d-1} \geq \frac{i}{d-1}\cdot\frac{1}{|t|}\left(1 - \frac{a}{|t|^\alpha}\right).$$

Because of our choices of coordinates on U_p, in the open set $g_p \leq -t$, we have that, $\omega^{d-1} \geq \lambda^{d-1}\cdot\left(1 + O_\omega(\frac{1}{|t|^{1/(d-1)}})\right)$, where the involved constant depends only on ω. Consequently

$$\int_{-R}^{-\tilde{r}} dt \int_{div(s)_{g_p}(t)} \omega^{d-1} \geq \int_{-R}^{-\tilde{r}} dt \left(1 + O_\omega(\frac{1}{|t|^{1/(d-1)}})\right)\int_{div(s)_{g_p}(t)} \lambda^{d-1}$$
$$\geq \int_{-R}^{-\tilde{r}} \frac{i}{d-1}\cdot\frac{1}{|t|}\cdot\left(1 + O_\omega(\frac{1}{|t|^{1/(d-1)}})\right)\cdot\left(1 - \frac{a}{|t|^\alpha}\right)dt.$$

The lemma follows.

We now prove the Theorem: By 5.11 we have

$$T_{c_1(L)}(r) = \lim_{R\to\infty}\int_{-R}^{\log(r)} dt \int_{B_{g_p}(t)} c_1(L)\wedge\omega^{d-1} =$$

$$\lim_{R\to\infty}\int_{-R}^{\log(r)} dt \int_{div(s)_{g_p}(t)} \omega^{d-1} - \mu_p(\log\|s\|^2)(r) + \mu_p(\log\|s\|^2)(e^{-R}).$$

By definition, of the norm on the strict transform we have that

$$\mu_p(\log\|s\|)(e^{-R}) = \int_{g_p=-R} \log\|\tilde{s}\|^2 d^c g_p \wedge \omega^{d-1} - \frac{i}{d-1}\log(R).$$

We now notice that

$$\int_{-R}^{\log(r)} dt \int_{div(s)_{g_p}(t)} \omega^{d-1} = \int_{-R}^{-\tilde{r}} dt \int_{div(s)_{g_p}(t)} \omega^{d-1} + \int_{-\tilde{r}}^{\log(r)} dt \int_{div(s)_{g_p}(t)} \omega^{d-1};$$

consequently, if we let R go to infinity and we apply the previous lemma, the conclusion follows.

5.16 *Algebraic points on images of maps of finite order.*

Let (A,ω,g_p) be a conformally parabolic variety of dimension d with its Khäler form and Evans kernel.

Suppose that K is a number field (with a fixed embedding in $\mathbb{C}$ as usual) and X_K is a smooth projective variety of dimension N defined over K. We fix an hermitian ample line bundle L on X_K and models etc. as in §2.

Let $S \subset X_K(\overline{K})$ and α be a real number.

Let $\gamma : A \to X_K$ be an analytic map of finite order ρ. For every point $P \in S\cap\gamma(A)$ we suppose that the germ of $\gamma(A)$ is an LG germ of type α defined over the $K(P)$.

In this section we will prove that, under the condition above, we can construct a closed positive T current on A with finite mass and having Lelong number on each point of $\gamma^{-1}(S)$, bigger then one. If one imagine this T as the current associated to an analytic divisor on A, the fact that its have finite mass is very similar to an "algebraicity" condition while the condition on the Lelong numbers corresponds to the fact that the divisor passes trough $\gamma^{-1}(S)$. We will see that when A is quasi projective, this will imply that $\gamma^{-1}(S)$ is not Zariski dense.

We will suppose that the hypotheses above are fixed once for all.

Let $S' \subset A$ be a countable set.

If T is a current over A and $P \in A$, we will denote by $\nu(T,P)$ the Lelong number of T in P.

5.16 Definition. *We will denote by $\Omega(S')$ the real number*

$$\inf\left\{\int_A T\wedge\omega^{d-1} \;/\; T \text{ is a current of bidegree } (1;1) \text{ with } \nu(T,P)\geq 1\ \forall P\in S'\right\}.$$

More generally, if U is an open set of A, We will denote by $\Omega(S';U)$ the real number

$$\inf\left\{\int_U T\wedge\omega^{d-1} \ / \ T \text{ is a current on A of bidegree } (1;1) \text{ with } \nu(T,P)\geq 1\ \forall P\in S'\right\}.$$

A priori, $\Omega(S')$ may be infinite. The aim of this section is to show that, in the arithmetic situation, it is a finite number.

We give some tools to compute $\Omega(S')$.

5.17 Proposition. *Suppose that $S'=\bigcup_i S_i$ with $S_i\subseteq S_{i+1}$ and there exists a constant M such that $\Omega(S_i)\leq M$ for every i, then*

$$\Omega(S')\leq M.$$

Proof: For every positive ϵ, we can find currents T_i such that $\int_A T_i\wedge\omega^{d-1}\leq M$ and $\nu(T_i;P)\geq 1$ for every $P\in S_i$. By Ascoli–Arzela' Theorem we can find a subsequence T_{i_k} of the T_i converging to a current T. By construction $\int_A T\wedge\omega^{d-1}\leq M$ and, since $\bigcup_{i_k} S_{i_k}=S'$, for every $P\in S$, we have $\nu(T;P)\geq 1$. The conclusion follows.

The proposition above is useful because it allows to suppose that S' has finite cardinality.

Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of relatively compact open sets of A such that

– $\bigcup_n U_n=A$;

– $U_n\subset U_{n+1}$.

We call such a sequence *an exhausting sequence.*

A proposition similar to 5.17 allows to work with each term of an exhausting sequence.

5.18 Proposition. *Suppose that $\{U_n\}_{n\in\mathbb{N}}$ is an exhausting sequence and that there exists a constant M such that, for every $n\in\mathbb{N}$ we have $\Omega(S',U_n)\leq A$, then*

$$\Omega(S')\leq M.$$

Proof: For every positive ϵ, and index n, we can find a current T_n such that $\int_{U_n} T_n\wedge\omega^{n-1}\leq M+\epsilon$ and $\nu(T_n,P)\geq 1$ for every $P\in S$. By Ascoli–Arzela' theorem, we can find a subsequence $\{T_{1,1},T_{1,2},\ldots\}$ converging to a current T^1 on U_1 such that $\int_{U_1} T^1\wedge\omega^{d-1}\leq M+\epsilon$ and $\nu(T;P)\geq 1$ for every $P\in S'\cap U_1$. We can extract from the subsequence above a subsequence $\{T_{2,1},T_{2,2},\ldots\}$ converging to a current T^2 with the same properties on U_2 and so on. The sequence $\{T_{n,n}\}$ converges to a current T with $\int_A T^1\wedge\omega^{d-1}\leq M+\epsilon$ and $\nu(T;P)\geq 1$ for every $P\in S'$. The conclusion follows.

In the sequel, we fix an exhausting sequence $\{U_n\}_{n\in\mathbb{N}}$. Observe that such a sequence exist: it suffices to fix a point $p\in A$ and take $U_n:=B_{g_p}(n)$.

Let X be a projective variety and L be an ample line bundle on X equipped with a positive metric (we also suppose that a metric is fixed on X). Let $\gamma\colon A\to X$ be a map of finite order ρ with Zariski dense image. Let $S\subset A(\mathbb{C})$.

Consider the linear map

$$\gamma_D : H^0(X, L^D) \hookrightarrow H^0(A, \gamma^*(L^D));$$

it is injective because the image of γ is Zariski dense.

For every point $P \in S$, we denote by I_P the ideal sheaf of P in A. Given a finite subset $F \subset S$ we denote by E^i_D the kernel of the map obtained by composing γ_D with the restriction map $H^0(A, \gamma^*(L^D)) \to \bigoplus_{P\in F}(\mathcal{O}_A/I^i_P)\otimes L^D$ and by γ^i_D the induced map

$$\gamma^i_D : E^i_D \longrightarrow \bigoplus_{P\in F} S^i(T_P A^*) \otimes L^D_P.$$

Eventually, for every $P \in F$ we denote by $\gamma^i_{D,P}$ the composite of γ^i_D with the canonical projection $\bigoplus_{P\in F} S^i(T_P A^*) \otimes L^D_P \to S^i(T_P A^*) \otimes L^D_P$. We fix such a finite set F.

Since we fixed the metric ω on A, all the vector spaces involved are equipped with an hermitian metric.

5.19 Theorem. *Let $\epsilon > 0$. Suppose that we are in the situation above and $U \subset A$ is a relatively compact open set. Then There exists a constant C independent on i and D such that, for $\frac{i}{D} \gg 0$ we have*

$$\log \|\gamma^i_{D,P}\| \leq -i\frac{\Omega(F,U)}{\rho+\epsilon}\log\frac{i}{D} + C(i+D).$$

5.20 Remark. One should compare this statement with Theorem 4.15.

Proof: We fix $t_0 \in \mathbb{R}$ such that $B_{g_P}(t_0) \supset U$. Let $s \in E^i_D$, then the current $T_s := \frac{[div(\gamma^*(s))]}{i}$ over A is closed, positive of bidegree $(1,1)$ and $\nu(T_s, Q) \geq 1$ for every $Q \in F$. Thus

$$\int_U T_s \wedge \omega^{d-1} = \frac{1}{i}\int_{U\cap div(\gamma^*(s))} \omega^{d-1} \geq \Omega(F,U).$$

Let $\tilde{r}$ as in 5.13. For $\log(r) \geq t_0$ we have

$$\begin{aligned}
&\int_{-\tilde{r}}^{\log(r)} dt \int_{div(\gamma^*(s))_{g_p}(t)} \omega^{d-1} \\
&= \int_{-\tilde{r}}^{t_0} dt \int_{div(\gamma^*(s))_{g_p}(t)} \omega^{d-1} + \int_{t_0}^{\log(r)} dt \int_{div(\gamma^*(s))_{g_p}(t)} \omega^{d-1} \\
&\geq i\Omega(F,U)\left(\log(r) - t_0\right).
\end{aligned}$$

Consequently, by theorem 5.13 and formula 5.10.1 we find a constant, independent on s, i and D such that

$$\log \|s\|^2_\infty - i\Omega(F,U)\log(r) + D\cdot T_\gamma(r) \geq \log \|j^i(s)\|^2_{i,D}(P) + C(i+D).$$

Since γ is of finite order ρ, for every $\epsilon > 0$ we can find λ such that, for r sufficiently big, $T_\gamma(r) \leq \lambda r^{\rho+\epsilon}$. The function $-i\Omega(F,U)\log(r) + D\lambda r^{\rho+\epsilon}$ has a minimum when $r^{\rho+\epsilon} = i\frac{\Omega(F;U)}{D\lambda(\rho+\epsilon)}$; thus there is a constant C_1 such that

$$\log \|s\|_\infty^2 - i\frac{\Omega(F;U)}{\rho+\epsilon}\log\frac{i}{D} \geq \log\|j^i(s)\|_{i,D}^2(P) + C_1(i+D).$$

The conclusion follows.

Suppose that, as before, K is a number field and $\sigma : K \hookrightarrow \mathbb{C}$ is an inclusion. We fix an algebraic closure $\overline{K}$ of K in $\mathbb{C}$. As in the previous section, We fix a smooth quasi projective variety X_K over K and $S \subseteq X_K(\overline{K})$. For every positive integer r denote by S_r the set $\{x \in S \ / \ [\mathbb{Q}(x) : K] \leq r\}$.

The main theorem of this section is the following.

5.21 Theorem. *Suppose that (A,ω,g) is a conformally parabolic variety of dimension d. Let $\gamma : A \to X_K(\mathbb{C})$ be an analytic map of finite order ρ with Zariski dense image. Suppose that, for every $P \in S \cap \gamma(A)$, the formal germ $\hat{A}_P$ of $\gamma(A)$ near P is (the pull back) of an LG–germ of type α in its field of definition. Then, for every r,*

$$\frac{\Omega(\gamma^{-1}(S_r))}{r} \leq \frac{N+1}{N-d}\rho\alpha[K:\mathbb{Q}].$$

Proof: The proof is, *mutatis mutandis* identical to the proof of theorem 4.16. We give some details. By propositions 5.17 and 5.18 it suffices to prove the following: Given a finite subset F of $\gamma^{-1}(S_r)$, a relatively compact set $U \subset A$ and $\epsilon > 0$ we have that

$$\frac{\Omega(F,U)}{r} \leq \frac{N+1}{N-d}\alpha(\rho+\epsilon)[K:\mathbb{Q}].$$

Let F be such a set and L be a finite extension of K which contains $\mathbb{Q}(\gamma(P_j))$ for every $P_j \in F$.

We fix $P_j \in F$ and, and we fix an embedding $\sigma_{P_j} \in S_{\mathbb{Q}(P_j)}$ extending σ_0. If $i/D \gg 0$, from Corollary 3.13 and Theorem 5.19 we get,

$$\begin{aligned}
&\frac{1}{[\mathbb{Q}(P_j):\mathbb{Q}]}\sum_{\sigma\in M_{\mathbb{Q}(P_j)}}\log\|\gamma^i_{D,j}\|_\sigma \\
&\qquad = \frac{1}{[\mathbb{Q}(P_j):\mathbb{Q}]}\left(\log\|\gamma^i_{D,j}\|_{\sigma_{\sigma_{P_j}}} + \sum_{\sigma\neq\sigma_{P_j}}\log\|\gamma^i_{D,j}\|_\sigma\right) \\
&\qquad \leq -\frac{\Omega(F;U)}{(\rho+\epsilon)\cdot[\mathbb{Q}(P_j):\mathbb{Q}]}i\log\left(\frac{i}{D}\right) + \alpha\cdot i\log(i) + C_2\,(i+D)),
\end{aligned}$$

with C_2 independent on i, D and $\mathbb{Q}(P_j)$. Thus, we deduce

$$\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma\in M_L}\log\|\gamma_D^i\|_\sigma \le \frac{1}{[L:\mathbb{Q}]}\sum_{\sigma\in M_L}\sup_{P_j\in F}\log\|\gamma_{D,j}^i\|_\sigma + \log(Card(F))$$
$$\le -\frac{\Omega(F;U)}{r\cdot[K:\mathbb{Q}](\rho+\epsilon)} i\log\left(\frac{i}{D}\right) + \alpha\cdot i\log(i) + C_2\,(i+D) + \log(Card(F)).$$

But since the image is Zariski dense, we can apply Theorem 2.3 and we obtain that

$$\frac{\Omega(F;U)}{r\cdot[K:\mathbb{Q}]} \le \frac{N+1}{N-d}(\rho+\epsilon)\alpha.$$

The conclusion follows.

By taking weak limits of currents one obtain:

5.22 Theorem. *Suppose that we are in the hypotheses as above. Then there exists a closed positive current T of bidegree $(1,1)$ over A such that:*
– $\int_A T\wedge\omega^{d-1} \le 2\frac{N+1}{N-d}\rho\alpha[K:\mathbb{Q}]$;
– *for every $P\in\gamma^{-1}(S)$ we have that $\nu(T;P)\ge\frac{1}{[K(\gamma(P)):K]}$.*

Proof: By theorem 5.21, for every integer r and $\epsilon>0$ we can find a current T_r such that $\frac{1}{r}\int_A T_r\wedge\omega^{d-1} \le \frac{N+1}{N-d}\rho\alpha[K:\mathbb{Q}]+\epsilon$ and $\nu(T_r;P)\ge 1$ for every $P\in\gamma^{-1}(S_r)$.

For every positive integer R consider the current $T_R := \sum_{r=1}^{R}\frac{T_r}{r^2}$.

By construction $\int_A T_R\wedge\omega^{d-1} \le \left(\sum_{r=0}^{R}\frac{1}{r^2}\right)\left(\frac{N+1}{N-d}\rho\alpha[K:\mathbb{Q}]+\epsilon\right)$. Thus we may find a subsequence of the $\{T_R\}$ converging to a current T such that $\int_A T\wedge\omega^{d-1} \le \left(\sum_{r=0}^{\infty}\frac{1}{r^2}\right)\left(\frac{N+1}{N-d}\rho\alpha[K:\mathbb{Q}]+\epsilon\right) \le 2\left(\frac{N+1}{N-d}\rho\alpha[K:\mathbb{Q}]+\epsilon\right)$.

Let $P\in\gamma^{-1}(S_r)\setminus\gamma^{-1}(S_{r-1})$. For every $i\ge r$, the current $\frac{T_i}{i^2}$ has Lelong number on P bigger or equal then $\frac{1}{i^2}$. Consequently the Lelong number of T on P is bigger or equal then $\sum_{i=r}^{\infty}\frac{1}{i^2}$. Since $\sum_{i=r}^{\infty}\frac{1}{i^2}\ge\int_r^{\infty}\frac{1}{t^2}dt=\frac{1}{r}$. the conclusion follows

5.23 Remark. One can obtain as corollary a statement analogous to 4.19 (but one needs to change a little bit the proof). However it will be again a consequence of [Bo], thus we leave the details to the reader.

5.25 *Rational points on affine varieties.* In this subsection we will show some consequence of our theory for affine varieties.

Often one has an analytic map between an affine and a projective variety and one want to show that, if the map is not algebraic, then the preimage of the algebraic points is not Zariski dense under some condition. We will show that theorem 5.21 will easily imply this for maps of finite order (and LG germs). One should notice that theorem 5.21 is more general and may have other applications.

Before we prove the theorem, we want to show that the value distribution theory we developed here is essentially equivalent to the value distribution theory developed by Griffiths and King in [GK].

We begin by fixing the hypotheses and the notations as in [GK].

Let A be a smooth affine variety of dimension d defined over $\mathbb{C}$. We can embed A as a Zariski open set $A \hookrightarrow \overline{A}$, where $\overline{A}$ is a smooth projective variety. Moreover we can suppose that the closed set at infinity, $D_\infty := \overline{A} \setminus A$, is a divisor whose support is a divisor with simple normal crossing.

We fix such a compactification of A and a projective embedding $\iota\colon \overline{A} \hookrightarrow \mathbb{P}^M$ for a suitable M. We can suppose that there is an hyperplane H_∞ in $\mathbb{P}^M$ such that $D_\infty = H_\infty \cap \overline{A}$ (without multiplicity). We will denote by R the degree of $\overline{A}$ in $\mathbb{P}^M$. We will denote by $\mathcal{O}_{\overline{A}}(1)$ the restriction of the ample line bundle $\mathcal{O}_{\mathbb{P}^M}(1)$ to $\overline{A}$; we will suppose that it is equipped with the Fubini–Study metric $\|\cdot\|_{FS}$. To simplify notations we will suppose that $H^i(\overline{A}; \mathcal{O}_{\overline{A}}(1)) = 0$ for every $i > 0$ (this hypothesis is not indispensable, nevertheless it can always be supposed and simplify the statement of the main result).

Let ω be the standard Kähler form on $\overline{A}$ induced from the embedding ι.

Let $p \in A$ be a point.

We will now recall the classical definition of counting function, given for instance in [GK].

Choose a linear subspace $L \simeq \mathbb{P}^{M-(d+1)} \subset H_\infty$ which do not meet $\overline{A}$ and we project from L on a suitable projective space $\mathbb{P}(V) \simeq \mathbb{P}^d$.

We get a commutative diagram

$$\begin{array}{ccccc}
\overline{A} & \xrightarrow{\iota} & \mathbb{P}^M \setminus L & \longleftarrow & H_\infty \setminus L \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & \downarrow \\
\mathbb{P}^d & = & \mathbb{P}^d & \longleftarrow & \mathbb{P}^{d-1}
\end{array}$$

with $\pi^{-1}(\mathbb{P}^{d-1}) \cap \overline{A} = H_\infty$. From this we obtain a finite branched covering $\pi\colon A \to \mathbb{C}^d$. Denote by $Ram(\pi)$ the ramification divisor of π. We suppose the situation above fixed once for all.

We fix some coordinate on $\mathbb{C}^d$ and we define $\varphi := \pi^*(\log \|z\|^2)$, where, for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we define $\|z\|^2 := \sum_i |z|^2$.

If α is a $(1,1)$ form on A we define the Nevanlinna counting function with respect to ω to be

$$\begin{aligned}
\tilde{T}_\alpha(r) &:= \int_0^r \frac{dt}{t} \int_{B_\varphi(\log(t))} \alpha \wedge (dd^c\varphi)^{d-1} \\
&= \int_{-\infty}^{\log(r)} dt \int_{\varphi \le t} \alpha \wedge (dd^c\varphi)^{d-1} \\
&= \int_A (\log(r) - \varphi)^+ (\alpha \wedge dd^c\varphi)^{d-1}
\end{aligned}$$

(where, as in §4, $(f)^+$ means $\sup\{f, 0\}$).

Let X be projective variety and $\gamma\colon A \to X$ be an analytic map. Suppose that $\overline{L}$ is an

ample line bundle equipped with a positive metric. We define $\tilde{T}_\gamma(r)$ to be $\tilde{T}_{\gamma^*(c_1(\overline{L}))}(r)$. In [GK] chapter 2 many properties of $\tilde{T}_\gamma(r)$ are proved (in particular the fact that it is, up to bonded functions, independent on the chosen metric).

5.25 Definition. *We say that $\gamma\colon A \to X$ is of finite Nevanlinna order $\tilde{\rho}$ if*

$$\limsup_{r\to\infty} \frac{\log(\tilde{T}_\gamma(r))}{\log(r)} = \tilde{\rho}$$

One sees that, when A is an affine variety, we have two possible value distribution theory on it. We will show that these two theories are comparable, namely the two definitions of order of an analytic map coincide:

5.26 Proposition. *If $\gamma : A \to X$ is an analytic map then*

$$\rho = \tilde{\rho}.$$

We first need three lemmas. Although they are standard and easy, we provide a proof for reader's convenience.

If $L\colon \mathbb{R} \to \mathbb{R}$ is a positive increasing function, we will denote by ρ_L the number $\limsup_{r\to\infty} \frac{\log(L(r))}{\log(r)}$.

5.27 Lemma. *Let E be a subset of $\mathbb{R}$ having finite Lebesgue measure. Then, for R sufficiently big,*

$$\rho_L = \limsup_{r\geq R, r\notin E} \frac{\log(L(r))}{\log(r)} := \rho_E.$$

Proof: It is evident that $\rho_L \geq \rho_E$. Suppose that $\rho_L \geq \rho_E + \epsilon$. Let $S \gg 0$ such that $meas(E \cap [S;\infty]) \leq \delta$ (we will fix delta at the end of the proof). There are infinitely many $r \geq S$ such that $\log(L(r)) > (\rho_E + \epsilon)\log(r)$. Fix one of them. Then $(r, r+\delta) \not\subseteq E$, thus there exists $s \in (r, r+\delta) \setminus E$. Consequently

$$(\rho_E + \epsilon)\log(r) \leq \log(L(r)) \leq \log(L(s)) \leq (\rho_E + \epsilon/2)\log(s) \leq (\rho_E + \epsilon/2)(\log(r) + O(\frac{\delta}{r}))$$

and this is a contradiction.

5.28 Lemma. *(Lang) Let ψ be a positive function such that $\int_{1+\epsilon}^{\infty} \frac{1}{x\psi(x)}dx < \infty$. Let T be positive increasing function. Then there exists a subset $E \subset \mathbb{R}$ with $meas(E) \leq \int_{1+\epsilon}^{\infty} \frac{1}{x\psi(x)}dx < \infty$, such that, for every $x \notin E$,*

$$T'(x) \leq T(x)\psi(T(x)).$$

Proof: Consider the subset $F := \{x \text{ s.t } T'(x) \geq T(x)\psi(T(x))\}$. If we denote by $\mathbb{I}_F(x)$ the characteristic function of F, then one easily see that

$$\mathbb{I}_F(x) \leq \frac{T'(x)}{T(x)\psi(T(x))}.$$

Integrating the inequality above and making the change of variables $T(x) = y$ the conclusion follows.

5.29 Corollary. *Suppose that T is an increasing derivable function, then*

$$\rho_{T'} \leq \rho_T.$$

Proof: Take $\psi(x) = \log^{1+\epsilon}(x)$ and apply the two lemmas.

5.30 Lemma. *Let ω be the Kähler form on $\overline{A}$ then, on $\overline{A} \setminus \pi^{-1}(0)$,*

$$2\omega \geq dd^c(\varphi).$$

Proof: Since everything is invariant under the action of the unitary group $U(d+1) \times U(M-d)$, it suffices to verify the inequality on the point with homogeneus coordinates $[1, 0, \ldots, 0, 1, 0, \ldots, 0]$ (the first sequence is a sequence of d zeros). On that point an explicit calculation gives the result.

We can now start the proof of 5.26.

Proof: (of 5.26). By 5.30 one easily see that $\rho \geq \tilde{\rho}$. We need to prove the converse inequality.

Take $a \in \mathbb{R}$ sufficiently small, denote by $T^1_\gamma(r)$ (resp. $\tilde{T}^1_\gamma(r)$) the function

$$\int_a^{\log(r)} dt \int_{a \leq g_p \leq t} c_1(L) \wedge \omega^{d-1}$$

(resp. $\int_a^{\log(r)} dt \int_{a \leq \varphi \leq t} c_1(L) \wedge (dd^c(\varphi)^{d-1})$ then $\rho = \limsup \frac{\log(T^1_\gamma(r))}{\log(r)}$ (resp. $\tilde{\rho} = \ldots$).

Moreover, denoting by $T^2_\gamma(r)$ the function

$$\int_a^{\log(r)} dt \int_{a \leq \varphi \leq t} c_1(L) \wedge \omega^{n-1},$$

we have $\rho = \limsup \frac{\log(T^2_\gamma(r))}{\log(r)}$. Indeed, there exists a C^∞ function f on $A \setminus B_{g_p}(a)$ such that $g_p = \frac{1}{\deg(\overline{A})}\varphi + f$; indeed, on $A \setminus B_{g_p}(a)$, we have that $dd^c(\varphi) = -\delta_{H_\infty} + \alpha$, where α is a closed positive (1,1) form; thus $dd^c(g_p - \varphi) \wedge \omega^{d-1}$ is a C^∞ form.

Denote by θ the positive (1,1) form $dd^c(\varphi)$ and

$$T^p(r) := \int_a^{\log(r)} dt \int_{a \leq \varphi \leq t} c_1(L) \wedge \theta^{(n-1)-p} \wedge \omega^p.$$

By induction, it suffices to prove that $\rho_{T^p} = \rho_{T^{p-1}}$.

It is evident that $\rho_{T^p} \geq \rho_{T^{p-1}}$.

Denote the coordinates on $\mathbb{C}^d$ by $(z_1, \ldots, z_d)$ and by $u(z)$ the function $\pi^*(\sum |z_i|^2)$. A simple, computation, obtained by taking successive projections on hyperplanes of codimension one, shows that $\omega - \theta = dd^c(k)$ where k is the function $\log(1 + \frac{1}{h} + k_1)$, where k_1 is a function which is bounded and extends to a C^∞ function on $\overline{A} \setminus B_{g_p}(a)$. Observe that, consequently, k is bounded on $A \setminus B_{g_p}(a)$.

We now compute, using Stokes theorem,

$$T^{p+1}(r) - T^p(r) = \int_a^{\log(r)} dt \int_{a \leq \varphi \leq t} c_1(L) \wedge (\theta^{(d-1)-(p+1)} \wedge \omega^{p+1}) - \theta^{(d-1)-p} \wedge \omega^p$$

$$= \int_a^{\log(r)} dt \int_{a \leq \varphi \leq t} c_1(L) \wedge \theta^{(d-1)-(p+1)} \wedge \omega^p \wedge (dd^c(k))$$

$$= \int_{\varphi = \log(r)} kd^c(\varphi) \wedge \theta^{(d-1)-(p+1)} \wedge \omega^p - \int_{a \leq \varphi \leq \log(r)} kdd^c(\varphi) \wedge \theta^{(d-1)-(p+1)} \wedge \omega^p + C.$$

But, since k is bounded, we apply Stokes theorem again and find a constant C such that

$$T^{p+1}(r) - T^p(r) \leq C \cdot \int_{a \leq \varphi \leq \log(r)} c_1(F) \wedge \theta^{(d-1)-p} \wedge \omega^p = \frac{dT^p}{dr}(r).$$

The conclusion follows from 5.29.

5.31 Definition. *Let $S \subset A(\mathbb{C})$; we will say that $\varpi(S) = m$ if the natural map*

$$\beta_m : H^0(\overline{A}; \mathcal{O}_{\overline{A}}(m) \longrightarrow \prod_{p \in S} \mathcal{O}_{\overline{A}}(m)|_p$$

is not *injective but the map*

$$\beta_{m-1} \colon H^0(\overline{A}; \mathcal{O}_{\overline{A}}(m-1) \longrightarrow \prod_{p \in S} \mathcal{O}_{\overline{A}}(m-1)|_p$$

is injective.

5.32 Remark. a) If β_m is not injective, then, for every $\ell \geq 0$, $\beta_{m+\ell}$ is not injective.

b) If $\varpi(S) = m$ then S is contained in the pull back of a divisor of degree m on $\mathbb{P}^M$ and *is not* contained in any pull back of divisors of degree strictly less then m.

Let X_K be a projective variety of dimension N and $S \subseteq X_K(\overline{K})$; We will define S_r as before.

5.33 Theorem. *Let $\gamma : A \to X_K(\mathbb{C})$ be an analytic map of finite order ρ. Suppose that, for every $P \in S \cap \gamma(A)$, the germ of $\gamma(A)$ near P is isomorphic to an LG germ of*

type α in the field of definition of P. Then for every positive integer r we have

$$\frac{\varpi(\gamma^{-1}(S_r))}{r} \leq \frac{N+1}{N-d}\frac{d!\rho\cdot\alpha}{\deg(\overline{A})}.$$

As a consequence, for every r, $\gamma^{-1}(S_r)$ is *not* Zariski dense in A.

Proof: Denote by V_m the vector space $H^0(\overline{A}, \mathcal{O}_{\overline{A}}(m))$ and fix a basis $\{s_0; s_1; \ldots; s_\ell\}$ of V_1 such that $div(s_0) = D_\infty$. Suppose that $\varpi(\gamma^{-1}(S_r)) > m$. We can find a finite set $F \subset \gamma^{-1}(S_r)$ such that the map

$$\beta_m : V_m \longrightarrow \prod_{P\in F} \mathcal{O}_{\overline{A}}(m)|_P$$

is an isomorphism. Moreover, since $s_0(P) \neq 0$ for every $P \in F$, we have a commutative diagram

$$\begin{array}{ccc} V_m & \xrightarrow{\simeq} & \prod_{P\in F}\mathcal{O}_{\overline{A}}(m)|_P \\ \big\downarrow{\scriptstyle \cdot s_0} & & \simeq\big\downarrow{\scriptstyle \cdot s_0} \\ V_{m+1} & \longrightarrow & \prod_{P\in F}\mathcal{O}_{\overline{A}}(m+1)|_P. \end{array}$$

For every multindex $I = (i_0, i_1, \ldots, i_\ell)$ we denote by $|I|$ the sum $i_0 + i_1 + \ldots + i_\ell$ and by s^I the section $s_0^{i_0} s_1^{i_1} \cdots s_\ell^{i_\ell} \in V_{|I|}$. Because of the commutative diagram above, for every multindex I with $|I| = m+1$ and $i_0 = 0$ we can find an element $R_I \in V_m$ such that $\beta_{m+1}(s_I + R_I \cdot s_0) = 0$. We will denote by Q_I the section $s_I + R_I \cdot s_0$.

The linear system generated by the Q_I and s_0^{m+1} define a map $Q : \overline{A} \to \mathbb{P}^h$.

We claim that the map Q is defined everywhere and finite: indeed, it is defined everywhere because, s_0 do not vanish outside the hyperplane at infinity D_∞ and, the Q_I cut over this hyperplane the complete linear system $H^0(\mathcal{O}(m+1))$. It is finite: we first observe that the restriction of Q to the hyperplane at infinity is an embedding (it is the restriction of the $m+1$–th Veronese embedding). There is an hyperplane H in $\mathbb{P}^h$, the pull back of which is the hyperplane at infinity D_∞; if there was a fibre of positive dimension this would cut the hyperplane at infinity somewhere and this is not possible.

We have that $Q(F) = [0:0:\ldots:0:1]$.

Let $b : \tilde{A} \to \overline{A}$ be the blow up of $\overline{A}$ in $Q^*([0:0:\ldots:0:1])$ and E be the exceptional divisor. By projection, we have a map

$$\tilde{P} : \tilde{A} \longrightarrow \mathbb{P}^{h-1}$$

and $\tilde{p}^*(O(1)) = b^*(\mathcal{O}_{\overline{A}}(m)) - E$. Thus $b^*(\mathcal{O}_{\overline{A}}(m)) - E$ is a nef line bundle on $\tilde{A}$.

By theorem 5.21 we can find a closed positive current T on A of bidegree $(1,1)$ such that

– $\frac{1}{r}\int_A T \wedge \omega^{d-1} \leq \frac{N+1}{N-d}\rho\alpha[K : \mathbb{Q}]$;

– For every $P \in F$, we have $\nu(T, P) \geq 1$.

Let $U := A \setminus \{F\}$ and $\mathbb{I}_U$ its characteristic function. By Skoda–El Mir Theorem, the current $\mathbb{I}_U T$ extends to a closed positive current $\tilde{T}$ on $\tilde{A}$. By [GK] lemma 1.16 and

prop. 1.17 (to be precise by a small variation of them, cf. remark after lemma 1.16 of loc cit.), and the fact that $\nu(T,P)\geq 1$ for every $P\in F$ we obtain

$$(\tilde{T};E^{d-1})\geq Card(F).$$

Since $\tilde{T}$ is closed and positive and $b^*(\mathcal{O}_{\overline{A}}(m))-E$ is nef,

$$(\tilde{T};(b^*(\mathcal{O}_{\overline{A}}(m))-E)^{d-1})=m^{d-1}(\tilde{T};(b^*(\mathcal{O}_{\overline{A}}(1)^{d-1})))-(\tilde{T};E^{d-1})\geq 0.$$

Thus

$$m^{d-1}r\frac{N+1}{N-d}\rho\alpha[K:\mathbb{Q}]\geq Card(F)=h^0(\overline{A};\mathcal{O}_{\overline{A}}(m))=\frac{m^d}{d!}\deg(\overline{A}).$$

The conclusion follows.

We remark the following corollary where the fact that Theorem 5.33 is a generalization of the Bombieri Schneider Lang criterion become evident:

5.34 Corollary. *Let X be a projective variety defined over a number field K and let $F\hookrightarrow T_X$ be a foliation. Suppose that $\gamma\colon A\to X$ is an analytic map of finite order ρ from an affine variety to X such that the image is a non algebraic leaf of the foliation. Then $\gamma^{-1}(X(K)\setminus Sing(F))$ is not Zariski dense.*

We find Bombieri Schneider Lang Theorem by taking $X=\mathbb{C}^N$ and $A=\mathbb{C}^n$.

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C. Gasbarri: Dipartimento di Matematica dell'Università di Roma "Tor Vergata", Viale della Ricerca Scientifica, 00133 Roma (I).

gasbarri@mat.uniroma2.it